14/2/23.

COMPLEX ANALYSIS AND LAPLACE TRANSFORM

UNIT I: VECTOR CALCULUS

Gradiant:

Let ϕ of (x, y, z) be any scalar point function, then

$$\nabla = \vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}$$

$\nabla\phi = \vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z}$ is called Gradiant of scalar point function.

$$\boxed{\text{grad}\,\phi = \nabla\phi.}$$

Problems on Gradiant

1) If $\phi = \log(x^2 + y^2 + z^2)$ then find $\nabla\phi$.

Soln:

W.K.T

$$\nabla\phi = \vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z}$$

Given:

$$\phi = \log(x^2 + y^2 + z^2)$$

$$\frac{\partial \phi}{\partial x} = \frac{2x}{(x^2+y^2+z^2)}$$

$$\frac{\partial \phi}{\partial y} = \frac{2y}{(x^2+y^2+z^2)}$$

$$\frac{\partial \phi}{\partial z} = \frac{2z}{(x^2+y^2+z^2)}$$

$$\nabla \phi = \vec{i}\frac{2x}{(x^2+y^2+z^2)} + \vec{j}\frac{2y}{(x^2+y^2+z^2)} + \vec{k}\frac{2z}{(x^2+y^2+z^2)}$$

$$= \frac{2}{(x^2+y^2+z^2)}[x\vec{i}+y\vec{j}+z\vec{k}].$$

$$\nabla \phi = \frac{2}{r^2}\vec{r} \quad \left[\because \vec{r} = x\vec{i}+y\vec{j}+z\vec{k},\ |\vec{r}| = \sqrt{x^2+y^2+z^2},\ r^2 = x^2+y^2+z^2\right]$$

Directional Derivative:

$$\text{Directional Derivative} = \frac{\nabla \phi \cdot \vec{a}}{|\vec{a}|}$$

2) To find the directional derivative of $\phi = x^2yz + 4xz^2$ at $(1,-2,1)$ in the direction of $2\vec{i} - \vec{j} - 2\vec{k}$.

$$D.D = \frac{\nabla\phi \cdot \vec{a}}{|\vec{a}|}$$

Given:

$$\phi = x^2yz + 4xz^2$$

$$\vec{a} = 2\vec{i} - \vec{j} - 2\vec{k}$$

$$|\vec{a}| = \sqrt{2^2 - (-1)^2 + (-2)^2} = \sqrt{9} = 3.$$

Diff ϕ w.r to x, y, z

$$\frac{\partial\phi}{\partial x} = 2xyz + 4z^2$$

$$\frac{\partial\phi}{\partial y} = x^2z$$

$$\frac{\partial\phi}{\partial z} = x^2y + 2xz$$

$$\nabla\phi = \vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z}$$

$$= \vec{i}(2xyz + 4xz^2) + \vec{j}(x^2z) + \vec{k}(x^2y + 8xz)$$

$$\nabla\phi_{(1,-2,-1)} = \vec{i}(8) - \vec{j} - 10\vec{k}$$

$$D.D = \frac{\nabla\phi \cdot \vec{a}}{|\vec{a}|}$$

$$= \frac{(8\vec{i} - \vec{j} - 10\vec{k}) \cdot (2\vec{i} - \vec{j} - 2\vec{k})}{3}$$

$$= \frac{16 + 1 + 20}{3}$$

$$= \frac{37}{3}$$

2M 3) Prove that $\nabla(r^n) = n r^{n-2} \vec{r}$

SCM

$$\vec{\nabla} = \sum \vec{i} \frac{\partial}{\partial x}$$

$$L.H.S = \nabla r^n$$

$$= \sum \vec{i} \frac{\partial}{\partial x}(r^n)$$

$$= \sum \vec{i} (n r^{n-1}) \frac{\partial r}{\partial x}$$

$$r = \sqrt{x^2 + y^2 + z^2}$$

$$\frac{\partial r}{\partial x} = \frac{2x}{2\sqrt{x^2+y^2+z^2}} = \frac{x}{r}$$

$$\nabla(r^n) = \sum \vec{i} (n r^{n-1}) \frac{x}{r}$$

$$= \sum \vec{i}\,(n r^{n-1 \cdot r})\, x$$

$$= \sum \vec{i}\,(n r^{n-2})\, x$$

$$= n r^{n-2} \sum x \vec{i}$$

$$= n r^{n-2} [x\vec{i} + y\vec{j} + z\vec{k}]$$

$$= n r^{n-2} \vec{r}$$

$= R.H.S.$

4) Find the directional derivative of $\phi = x^2yz + 4xz^2 + xyz$ at $(1, 2, 3)$ in the direction of $2\vec{i} + \vec{j} - \vec{k}$

H.W

Sm $\quad D.D = \dfrac{\nabla\phi \cdot \vec{a}}{|\vec{a}|}$

Given :

$$\phi = x^2yz + 4xz^2 + xyz$$

$$\vec{a} = 2\vec{i} + \vec{j} - \vec{k}$$

$$|\vec{a}| = \sqrt{(2)^2 + (1)^2 + (-1)^2} = \sqrt{6}.$$

Diff ϕ w.r to x, y, z

$$\frac{\partial\phi}{\partial x} = 2xyz + 4z^2 + yz$$

$$\frac{\partial\phi}{\partial y} = x^2z + xz$$

$$\frac{\partial\phi}{\partial z} = x^2y + 8xz + xy.$$

$$\nabla\phi = \vec{i}\,\frac{\partial\phi}{\partial x} + \vec{j}\,\frac{\partial\phi}{\partial y} + \vec{k}\,\frac{\partial\phi}{\partial z}$$

$$= \vec{i}\,(2xyz + 4z^2 + yz) + \vec{j}\,(x^2z + xz)$$

$$+ \vec{k}\,(x^2y + 8xz + xy).$$

$$\nabla\phi_{(1,2,3)} = \vec{i}\,[2(1)(2)(3) + 4(3)^2 + (2)(3$$

$$+ \vec{j}\,[(1)^2(3) + (1)(3)] + \vec{k}\,[(1)^2(2) + 8(1)(3$$

$$+ (1)(2)]$$

$$= \vec{i}\,[12 + 56 + 6] + \vec{j}\,[3+3] + \vec{k}\,[2 + 24 + 2$$

$$\nabla\phi = 54\vec{i} + 6\vec{j} + 28\vec{k}$$

$$D.D = \frac{\nabla\phi\cdot\vec{a}}{|\vec{a}|}$$

$$= \frac{54\vec{i} + 6\vec{j} + 28\vec{k}\cdot(2\vec{i} + \vec{j} - \vec{k})}{\sqrt{6}}$$

$$= \frac{108 + 6 - 28}{\sqrt{6}}$$

$$D.D. = \frac{86}{\sqrt{6}}$$

Unit Tangent Vector:-

$$\text{Unit tangent vector} = \frac{\frac{d\vec{r}}{dt}}{\left|\frac{d\vec{r}}{dt}\right|}.$$

1) Find the unit tangent vector to the following surfaces at the specified points $x = t^2+1$, $y = 4t-3$, $z = 2t^2 - 6t$ at $t = 2$.

Sln:

$$\text{Unit tangent vector} = \frac{d\vec{r}}{dt} \Big/ \left|\frac{d\vec{r}}{dt}\right|.$$

$$\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$$

$$= (t^2+1)\vec{i} + (4t-3)\vec{j} + (2t^2-6t)\vec{k}$$

$$\frac{d\vec{r}}{dt} = 2t\vec{i} + 4\vec{j} + (4t-6)\vec{k}$$

$$\left(\frac{d\vec{r}}{dt}\right)_{t=2} = 4\vec{i} + 4\vec{j} + 2\vec{k}$$

$$\left|\frac{d\vec{r}}{dt}\right| = \sqrt{4^2+4^2+2^2}$$

$$= \sqrt{36}$$

$$= 6$$

$$\text{Unit tangent vector} = \frac{4\vec{i} + 4\vec{j} + 2\vec{k}}{6}$$

$$= \frac{2(2\vec{i}+2\vec{j}+\vec{k})}{6}$$

$$\text{U.T.V} = \frac{2\vec{i}+2\vec{j}+\vec{k}}{3}$$

Unit Normal vector:

$$\hat{n} = \frac{\nabla\phi}{|\nabla\phi|}$$

1) Find the unit normal vector to the surface $x^2+y^2+z=11$ at $(1,1,1)$.

Soln: Given:

$$\phi = x^2+y^2-z-11$$

$$\nabla\phi = \vec{i}\frac{\partial\phi}{\partial x}+\vec{j}\frac{\partial\phi}{\partial y}+\vec{k}\frac{\partial\phi}{\partial z}$$

$$\frac{\partial\phi}{\partial x} = 2x, \quad \frac{\partial\phi}{\partial y} = 2y, \quad \frac{\partial\phi}{\partial z} = -1.$$

$$\nabla\phi = 2x\vec{i}+2y\vec{j}-1\vec{k}$$

$$\nabla\phi_{(1,1,1)} = 2\vec{i}+2\vec{j}-\vec{k}$$

$$|\nabla\phi| = \sqrt{2^2+2^2+(-1)^2}$$

$$= \sqrt{4+4+1} = \sqrt{9}$$

$$= 3$$

$$\hat{n} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{2\vec{i} + 2\vec{j} - \vec{k}}{3}.$$

Normal derivative :

$$\text{Normal derivative} = |\nabla\phi|$$

1) Find the normal derivative of $\phi = xy + yz + zx$ at $(-1, 1, 1)$.

soln: Given :

$$\phi = xy + yz + zx.$$

$$\nabla\phi = \vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z}$$

$$\frac{\partial\phi}{\partial x} = y + z \quad \frac{\partial\phi}{\partial y} = x + z \quad \frac{\partial\phi}{\partial z} = y + x$$

$$\nabla\phi = (y+z)\vec{i} + (x+z)\vec{j} + (y+x)\vec{k}.$$

$$\nabla\phi_{(-1,1,1)} = (2)\vec{i} + 0\vec{j} + 0\vec{k}$$

$$= 2\vec{i}$$

$$|\nabla\phi| = \sqrt{2^2}$$

$$= \sqrt{4}$$

$$= 2.$$

Angle between the Surfaces:

$$\cos\theta = \frac{\nabla\phi_1 \cdot \nabla\phi_2}{|\nabla\phi_1||\nabla\phi_2|}$$

$$\theta = \cos^{-1}\frac{\nabla\phi_1 \cdot \nabla\phi_2}{|\nabla\phi_1||\nabla\phi_2|}$$

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1) Find the angle between the surfaces $x\log z = y^2 - 1$, $x^2 y = 2 - z$ at $(1, 1, 1)$.

Given:

$$\phi_1 = x\log z - y^2 + 1 \qquad \phi_2 = x^2 y - 2 + z$$

$$\frac{\partial\phi_1}{\partial x} = \log z \qquad \frac{\partial\phi_2}{\partial x} = 2xy$$

$$\frac{\partial\phi_1}{\partial y} = -2y \qquad \frac{\partial\phi_2}{\partial y} = x^2$$

$$\frac{\partial\phi_1}{\partial z} = \frac{x}{z} \qquad \frac{\partial\phi_2}{\partial x} = 1.$$

$$\nabla\phi_1 = \vec{i}\frac{\partial\phi_1}{\partial x} + \vec{j}\frac{\partial\phi_1}{\partial y} + \vec{k}\frac{\partial\phi_1}{\partial z}$$

$$= \log z\,\vec{i} - 2y\,\vec{j} + \frac{x}{z}\vec{k}$$

$$\nabla\phi_2 = \vec{i}\frac{\partial\phi_2}{\partial x} + \vec{j}\frac{\partial\phi_2}{\partial y} + \vec{k}\frac{\partial\phi_2}{\partial z}$$

$$\nabla\phi_2 = 2xy\,\vec{i} + x^2\vec{j} + \vec{k}$$

$\nabla\phi_{(1,1,1)} = \log(1)\vec{i} - 2\vec{j} + 1\vec{k}$

$\nabla\phi_{(1,1,1)} = -2\vec{j} + \vec{k}$

$|\nabla\phi_1| = \sqrt{(-2)^2 + (1)^2} = \sqrt{5}$

$\nabla\phi_2(1,1,1) = 2\vec{i} + \vec{j} + \vec{k}$

$|\nabla\phi_2| = \sqrt{2^2 + 1^2 + 1^2} = \sqrt{6}$

$$\cos\theta = \frac{\nabla\phi_1 \cdot \nabla\phi_2}{|\nabla\phi_1| \cdot |\nabla\phi_2|}$$

$$= \frac{(0\vec{i} - 2\vec{j} + \vec{k}) \cdot (2\vec{i} + \vec{j} + \vec{k})}{\sqrt{5} \cdot \sqrt{6}}$$

$$= \frac{0-2+1}{\sqrt{30}} = \frac{-1}{\sqrt{30}}$$

$$\cos\theta = \frac{-1}{\sqrt{30}}$$

$$\theta = \cos^{-1}\left(-1/\sqrt{30}\right)$$

$$\theta = \cos^{-1}\left(1/\sqrt{30}\right).$$

Divergence & curl:

Let $\vec{F} = F_1\vec{i} + F_2\vec{j} + F_3\vec{k}$.

Then, $\operatorname{div}\vec{F} = \nabla \cdot \vec{F}$.

$$\operatorname{div}\vec{f} = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right) \cdot \left(f_1\vec{i} + f_2\vec{j} + f_3\vec{k}\right).$$

$$\operatorname{div}\vec{f} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}$$

and $\vec{f} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ f_1 & f_2 & f_3 \end{vmatrix}$

1) If $\vec{F} = x^2\vec{i} + y^2\vec{j} + z^2\vec{k}$ then find $\nabla \cdot \vec{f}$, $\nabla \times \vec{f}$.

Sln $\nabla \vec{F} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}$

Given :

$$\vec{F} = x^2\vec{i} + y^2\vec{j} + z^2\vec{k}$$

$\vec{F_1} = x^2$, $\vec{F_2} = y^2$, $\vec{F_3} = z^2$.

$\frac{\partial f_1}{\partial x} = 2x$, $\frac{\partial f_2}{\partial y} = 2y$, $\frac{\partial f_3}{\partial z} = 2z$

$\nabla \cdot \vec{F} = 2x + 2y + 2z$

$\nabla \cdot \vec{F} = 2(x+y+z)$

$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x^2 & y^2 & z^2 \end{vmatrix}$

$= \vec{i}\left(\frac{\partial}{\partial y}(z)^2 - \frac{\partial}{\partial z}(y^2)\right) - \vec{j}\left(\frac{\partial}{\partial x}(z^2) - \frac{\partial}{\partial z}(x^2)\right) + \vec{k}\left(\frac{\partial}{\partial x}(y^2) - \frac{\partial}{\partial y}(x^2)\right)$.

$= 0\vec{i} + 0\vec{j} + 0\vec{k}$

$\boxed{\nabla \times \vec{F} = 0.}$

Note

If $\nabla \cdot \vec{F} = 0$, then the vector is solenoidal.

If $\nabla \times \vec{F} = 0$, then the vector is irrotational.

1) Prove that the vector $\vec{F} = z\vec{i} + x\vec{j} + y\vec{k}$ is solenoidal.

Sln: To prove : $\nabla \cdot \vec{F} = 0$

$$\nabla \cdot \vec{F} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}$$

$$\vec{F} = z\vec{i} + x\vec{j} + y\vec{k}$$

$$\nabla \cdot \vec{f} = 0 + 0 + 0.$$

$$\nabla \cdot \vec{F} = 0.$$

Therefore, given $\vec{F}$ is solenoidal.

Hence Proved.

2) If $\vec{f} = (x+3y)\vec{i} + (y-2z)\vec{j} + (x+\lambda z)\vec{k}$ is solenoidal. Find λ.

Sln: Given :

$$\vec{F} = (x+3y)\vec{i} + (y-2z)\vec{j} + (x+\lambda z)\vec{k}$$

W.K.T : $\nabla \cdot \vec{F} = 0$.

$$\nabla \cdot \vec{F} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z} \rightarrow (1)$$

$f_1 = (x+3y)$ $\quad f_2 = (y-2z)$ $\quad f_3 = (x+\lambda z)$

$$\frac{\partial f_1}{\partial x} = 1 \qquad \frac{\partial f_2}{\partial y} = 1 \qquad \frac{\partial f_3}{\partial z} = \lambda. \rightarrow (2)$$

sub (2) in (1)

$$\nabla \cdot \vec{F} \Rightarrow 1 + 1 + \lambda = 0$$

$$0 = 2 + \lambda$$

$$\boxed{\lambda = -2.}$$

3) Find the constants a, b, c so that

Find $\vec{F} = (x + 2y + az)\vec{i} + (bx - 3y - z)\vec{j} + (4x + cy + 2z)\vec{k}$ is irrotational

Ans: $\nabla \times \vec{F} = 0.$

$$\vec{F} = yz\vec{i} + zx\vec{j} + xy\vec{k}$$

$\vec{F_1} = yz$ $\quad \vec{F_2} = zx$ $\quad \vec{F_3} = xy.$

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ (x+2y+az) & (bx-3y-z) & (4x+cy+2z) \end{vmatrix}$$

$$= \vec{i}\left(\frac{\partial}{\partial x}(bx-3y-z)\right.$$

$$= \vec{i}\left(\frac{\partial}{\partial y}(4x+cy+2z) - \frac{\partial}{\partial z}(bx-3y-z)\right)$$

$$- \vec{j}\left(\frac{\partial}{\partial x}(4x+cy+2z) - \frac{\partial}{\partial z}(x+2y+az)\right)$$

$$+ \vec{k}\left(\frac{\partial}{\partial x}(bx-3y-z) - \frac{\partial}{\partial y}(x+2y+az)\right)$$

$$= 0.$$

$$\vec{i}\left[(c)+1\right] - \vec{j}\left[(4)-a\right] + \vec{k}(b-2) = 0.$$

$$(c+1)\vec{i} - (4-a)\vec{j} + (b-2)\vec{k} = 0\vec{i} + 0\vec{j} + 0\vec{k}$$

$c+1=0$ $\quad -4+a=0 \quad$ $b-2=0$

$c=-1$ $\quad a=4 \quad$ $b=2.$

$\therefore a=4$

$b=2$

$c=-1.$

4) Prove that curl (grad ϕ) $=0$.

To Prove:

curl (grad ϕ) $=0$

grad $\phi = \nabla\phi$.

$$\nabla\phi = \vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z}$$

$$\text{curl}(\text{grad}\,\phi) = \nabla \times \nabla\phi$$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \partial\phi/\partial x & \partial\phi/\partial y & \partial\phi/\partial z \end{vmatrix}$$

$$= \vec{i}\left(\frac{\partial^2\phi}{\partial y\,\partial z} - \frac{\partial^2\phi}{\partial z\,\partial y}\right) - \vec{j}\left(\frac{\partial^2\phi}{\partial x\,\partial z} - \frac{\partial^2\phi}{\partial z\,\partial x}\right)$$

$$+ \vec{k}\left(\frac{\partial^2\phi}{\partial x\,\partial y} - \frac{\partial^2\phi}{\partial y\,\partial x}\right).$$

$$= 0\vec{i} + 0\vec{j} + 0\vec{k}$$

Curl (grad ϕ) = 0

Hence Proved.

H.W Show that $\vec{F} = yz\vec{i} + zx\vec{j} + xy\vec{k}$ is irrational.

Sln: $\nabla \times \vec{F} = 0$

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ yz & zx & xy \end{vmatrix}$$

$$= \vec{i}\left(\frac{\partial}{\partial y}(xy) - \frac{\partial}{\partial z}(zx)\right) - \vec{j}\left(\frac{\partial}{\partial x}(xy) - \frac{\partial}{\partial z}(yz)\right)$$

$$+ \vec{k}\left(\frac{\partial}{\partial x}(zx) - \frac{\partial}{\partial y}(yz)\right).$$

$$= \vec{i}(x - x) - \vec{j}(y - y) + \vec{k}(z - z)$$

$$= 0\vec{i} - 0\vec{j} + 0\vec{k}$$

$$= 0.$$

$$\nabla \times \vec{F} = 0.$$

$\therefore \vec{F}$ is irrational

Hence Proved.

i) Prove that $\vec{F} = (y^2 \cos x + z^3)\vec{i} + (2y \sin x - 4)\vec{j} + 3xz^2\vec{k}$ is irrotational & find its scalar Potential.

Sln: $\vec{F} = (y^2 \cos x + z^3)\vec{i} + (2y\sin x - 4)\vec{j} + 3xz^2\vec{k}$.

$$\nabla \times \vec{F} = 0.$$

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y^2\cos x + z^3 & 2y\sin x - 4 & 3xz^2 \end{vmatrix}$$

$$= \vec{i}\left(\partial/\partial y\,(3x - z^2) - \frac{\partial}{\partial z}(2y\sin x - 4)\right) - \vec{j}\left(\frac{\partial}{\partial x}(3xz^2) - \frac{\partial}{\partial z}(y^2\cos x + z^3)\right) + \vec{k}\left(\frac{\partial}{\partial x}(2y\sin x - 4) - \frac{\partial}{\partial y}(y^2\cos x + z^3)\right)$$

$$= \vec{i}(0-0) - \vec{j}(3z^2 - 3z^2) + \vec{k}(2y\cos x - 2y\cos x)$$

$$= 0\vec{i} - 0\vec{j} + 0\vec{k}$$

$$= 0.$$

$$\nabla \times \vec{F} = 0.$$

$\therefore$ $\vec{F}$ is irrotational.

Hence Proved.

To find Scalar Potential(ϕ).

$\vec{F} = \nabla\phi$

$(y^2\cos x + z^3)\vec{i} + (2y\sin x - 4)\vec{j} +$

$(3xz^2)\vec{k} = \vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z}$.

Equating the both sides with co-eff $\vec{i}, \vec{j}, \vec{k}$.

$y^2\cos x + z^3 = \frac{\partial\phi}{\partial x}$ $\quad 2y\sin x - 4 = \frac{\partial\phi}{\partial y}$

Integrate both sides, $\quad 3xz^2 = \frac{\partial\phi}{\partial z}$

$\int(y^2\cos x + z^3)dx = \int\frac{\partial\phi}{\partial x}dx$, $\int(2y\sin x - 4)dy = \int\frac{\partial\phi}{\partial y}dy$

$\int 3xz^2\,dz = \int\frac{\partial\phi}{\partial z}dz$

$\phi = y^2\sin x + z^3x$ (1) $\quad \phi = y^2\sin x - 4y$ (2) $\quad \phi = xz^3$ (3)

combine (1), (2), (3)

$\boxed{\phi = y^2\sin x + z^3x - 4y.}$

3) Prove that $\vec{F} = (6xy + z^3)\vec{i} + (3x^2 - z)\vec{j} + (3xz^2 - y)\vec{k}$ is irrotational & find its scalar potential, such that $\vec{f} = \nabla\phi$.

Ans: $\vec{F} = (6xy + z^3)\vec{i} + (3x^2 - z)\vec{j} + (3xz^2 - y)\vec{k}$.

$\nabla \times \vec{F} = 0$.

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 6xy+z^3 & 3x^2-z & 3xz^2-y \end{vmatrix}$$

$$= \vec{i}\left(\partial/\partial y(3xz^2-y) - \frac{\partial}{\partial z}(3x^2-z)\right) - \vec{j}$$

$$\left(\frac{\partial}{\partial x}(3xz^2-y) - \frac{\partial}{\partial z}(6xy+z^3)\right) + \vec{k}$$

$$\left(\frac{\partial}{\partial x}(3x^2-z) - \frac{\partial}{\partial y}(6xy+z^3)\right).$$

$$= \vec{i}(-1+1) - \vec{j}(3z^2-3z^2) + \vec{k}(6x-6x)$$

$$= 0\vec{i} - 0\vec{j} + 0\vec{k}$$

$$= 0.$$

$$\nabla \times \vec{F} = 0.$$

$\therefore$ $\vec{F}$ is irrational.

Hence Proved.

$$\vec{F} = \nabla\phi$$

$$(6xy+z^3)\vec{i} + (3x^2-z)\vec{j} + (3xz^2-y)\vec{k} =$$

$$\vec{i}\left(\frac{\partial\phi}{\partial x}\right) + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z}$$

$$6xy+z^3 = \frac{\partial\phi}{\partial x}, \quad 3x^2-z = \frac{\partial\phi}{\partial y},$$

$$3xz^2-y = \frac{\partial\phi}{\partial z}$$

∫ing

$$\int(6xy\cancel{z}+z^3)\,dx = \int \frac{\partial \phi}{\partial x}\,dx, \quad \int(3x^2-z)\,dy = \int \frac{\partial \phi}{\partial y}\,dy$$

$$\int(3xz^2-y)\,dz = \int \frac{\partial \phi}{\partial z}\,dz$$

$$\frac{6yx^3}{3}+z^3=\phi \to (1), \quad 3x^2y-2y=\phi \to (2)$$

$$\frac{3xz^3}{3}-yz=\phi \to (3)$$

combining (1), (2), (3)

$$\phi = 3x^2+xz^3-2y$$