Green's theorem : If u, v $\frac{\partial u}{\partial y}$, $\frac{\partial v}{\partial x}$

If $u, v, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}$ are continous and single value functions of region ~~are~~ R enclosed by any curve c, then

$$\int_C u\,dx + v\,dy = \iint_R \left( \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) dx\,dy.$$

1) Verify greens theorem in the xy plane for $\int_C (3x - 8y^2)\,dx + (4y - 6xy)\,dy$, where $C$ is the boundary of the region given by $x = 0,\ y = 0,\ x + y = 1$.

~~(4y-6y)~~

Sln:

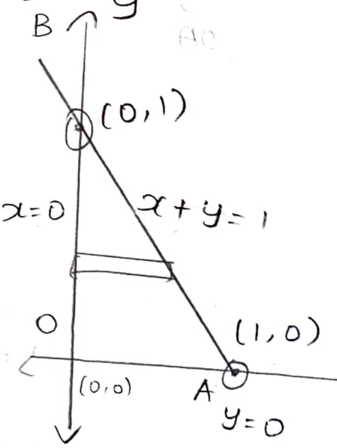
By Green's theorem:

$$\int_C u\,dx + v\,dy = \iint_R \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dx\,dy$$

$u = 3x - 8y^2$ $\qquad$ $v = 4y - 6xy$

$\frac{\partial u}{\partial y} = -16y$ $\qquad$ $\frac{\partial v}{\partial x} = -6y$

R.H.S:-




$$\iint_R \left[\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right] dx\,dy$$

$$= \iint (-6y + 16y)\,dx\,dy$$

$$= \int_0^1 \int_0^{1-y} (10y)\,dx\,dy$$

$$= \int_0^1 (10xy)_0^{1-y}\,dy$$

$$= \int_0^1 [10(1-y)y]\,dy$$

$$= \int_0^1 (10y - 10y^2)\,dy$$

$x = 0,\ x = 1 - y$

$y = 0,\ y = 1.$

$x + y = 1$

| | | |
|---|---|---|
| $x$ | 0 | 1 |
| $y$ | 1 | 0 |

$$= 10\left[\frac{y^2}{2} - \frac{y^3}{3}\right]_0^1$$

$$= 10\left[\frac{1}{2} - \frac{1}{3}\right] = 10\left[\frac{1}{6}\right]$$

$$= 5/3.$$

L.H.S

$$\int u\,dx + v\,dy = \int_{OA} + \int_{AB} + \int_{BO}$$

Along OA : $[y = 0,\ dy = 0]$.

$$\int_{OA} (3x - 8y^2)\,\overset{dx}{\cancel{dy}} + (4y - 6xy)\,dy =$$

$$\int_0^1 3x\,dx$$

$$= 3\left(\frac{x^2}{2}\right)_0^{\cancel{0}\,1}$$

$$= 3/2.$$

Along AB : $[y = 1 - x,\ dy = -dx]$

$$\int (3x - 8y^2)\,dx + (4y - 6xy)\,dy =$$

$$\int_1^0 [3x - 8(1-x)^2]\,dx +$$

$$[4(1-x) - 6x(1-x)][-dx].$$

$$= \int_0^1 3x - 8\left[1 + x^2 - 2x\right] dx + (4 - 4x - 6x + 6x^2)\,dx\,(-dx)$$

$$= \int_1^0 (3x - 8 - 8x^2 + 16x - 4 + 4x + 6x - 6x^2)\,dx.$$

$$= \int_1^0 (-14x^2) + 29x - 12)\,dx$$

$$= \left[-14\left(\frac{x^3}{3}\right) + 29\left(\frac{x^2}{2}\right) - 12x\right]_1^0$$

$$= \frac{-14}{3}(-1) + \frac{29}{2}(-1) - 12(-1).$$

$$= \frac{14}{3} - \frac{29}{2} + 12$$

$$= \frac{14 - 29 + 12}{6}$$

$$= \frac{13}{6}.$$

Along BO

Along BO $(x = 0, dx = 0)$.

$$\int_{BO} (3x - 8y^2)\,dx + (4y - 6xy)\,dy =$$

$$\int_1^0 4y\,dy = 4\left(\frac{y^2}{2}\right)_1^0 = \frac{4}{2}(-1)$$

$$= -2.$$

L.H.S $\int_C = \int_{OA} + \int_{AB} + \int_{BO}$

$$= \frac{3}{2} + \frac{13}{6} - 2$$

$$= \frac{10}{6}$$

$$= 5/3.$$

LHS = RHS

Hence Green's theorem is Verified

closed curve

Green's theorem is verified

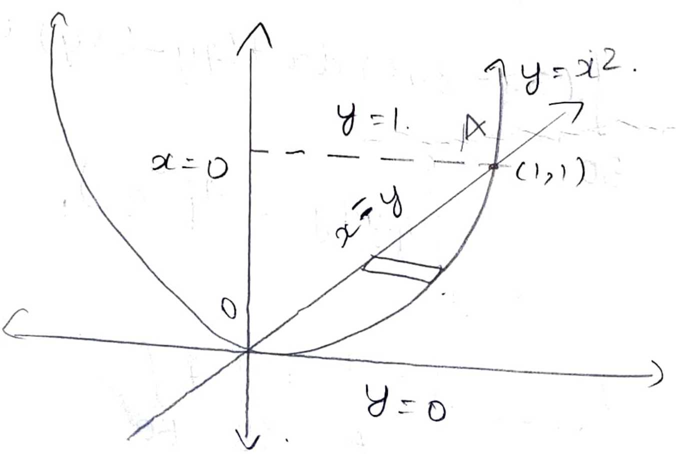
2) Verify Green's theorem in the XY Plane for $\int_C (xy + y^2)\, dx + x^2 dy$ where C is the closed curve of the region bounded by $y = x$ & $y = x^2$.

Sln: By Green's theorem:

$$\int_C u\,dx + v\,dy = \iint_R \left( \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) dx\,dy.$$

$OA = \frac{19}{20}$.

$AO = -1$



$u = xy + y^2 \qquad v = x^2$

$\frac{\partial u}{\partial y} = x + 2y \qquad \frac{\partial v}{\partial x} = 2x$

$y = x, \; y = x^2$

$y = x^2$

$x = x^2$

$x - x^2 = 0$

$x(1 - x) = 0$

$\boxed{x = 0, \; x = 1.}$

$y = y^2$

$y(1 - y) = 0$

$y = 0, \; y = 1.$

R.H.S

$$\iint_R \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dx\,dy =$$

$$\int_0^1 \int_y^{\sqrt{y}} \left[2x - (x + 2y)\right] dx\,dy$$

$$= \int_0^1 \int_y^{\sqrt{y}} (2x - x - 2y)\, dx\,dy$$

$\boxed{dx\,dy}$

$$= \int_0^1 \int_y^{\sqrt{y}} (x - 2y)\, dx\,dy$$

$$= \int_0^1 \left[\frac{x^2}{2} - 2xy\right]_y^{\sqrt{y}} dy$$

$$= \int_0^1 \left[\frac{y}{2} - 2y^{3/2} - \frac{y^2}{2} + 2y^2\right] dy.$$

$$= \left[\frac{y^2}{4} - 2\left(\frac{y^{5/2}}{5/2}\right) - \frac{y^3}{6} + 2\left(\frac{y^3}{3}\right)\right]_0^1$$

$$= \left[\frac{y^2}{4} + \frac{4}{5} y^{5/2} - \frac{y^3}{6} + \frac{2}{3} y^3\right]_0^1$$

$$= 1/4 - 4/5 - \frac{1}{6} + \frac{2}{3}$$

$$= \frac{-1}{20} \qquad = \frac{-1}{20}$$

L.H.S

$\int (xy + y^2)\,dx + (x^2)\,dy$

$\int u\,dx + v\,dy = \int_{OA} + \int_{AO}$

$\int_{OA}$ $\left[ y = x^2, \frac{dy}{dx} = 2x \Rightarrow dy = 2x\,dx \right].$

$\int (x^3 + x^4)\,dx + x^2\,2x\,dx$

$\int_0^1 (x^3 + x^4 + 2x^3)\,dx$

$\int_0^1 (3x^3 + x^4)\,dx$

$\left[ \frac{3}{4} x^4 + \cancel{x^4} \frac{1}{5} x^5 \right]_0^1$

$\frac{3}{4} + \frac{1}{5} = \frac{15 + 4}{20} = \frac{19}{20}$

$\int_{OA} = \frac{19}{20}$

$\int_{AO}$ $\left[ y = x, \frac{dy}{dx} = 1, \left[ dy = dx \right] \right.$

$\int (x^2 + x^2)\,dx + (x^2)\,dx$

$\int_1^0 3x^2 \cdot dx$

$\frac{3}{3} x^3 = -1.$

$\int u\,dx + v\,dy = -1 + \frac{19}{20}$

LHS $= -1/20$

Hence Green's theorem Proved.