Gauss Divergence Theorem

If V is the volume of the closed surfaces,

Then, $\iint_S \vec{F}\cdot\hat{n}\, ds = \iiint_V (\nabla\cdot\vec{F})\, dv$

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1) verify gauss divergence theorem for $\vec{F} = 4xz\vec{i} - y^2\vec{j} + yz\vec{k}$ over the cube x varies from 0 to 1 and y varies from 0 to 1 . z varies from 0 to 1.

Sol: By Gauss Divergence theorem

$$\iint_S \vec{F}\cdot\hat{n}\, ds = \iiint_V (\nabla\cdot\vec{F})\, dv$$

$(\nabla\cdot\vec{F})$

→ Gauss divergence theorem

R.H.S

$$\iiint_V (\nabla\cdot\vec{F})\, dv$$

Gauss

$\iiint_V (\nabla\cdot\vec{F})\, dv$

Given:

$$\vec{F} = 4xz\vec{i} - y^2\vec{j} + yz\vec{k}$$

$$\nabla\cdot\vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

$F_1 = 4xz \qquad F_2 = -y^2 \qquad F_3 = yz$

$$\frac{\partial F_1}{\partial x} = 4z \quad \frac{\partial F_2}{\partial y} = -2y \quad \frac{\partial F_3}{\partial z} = y$$

$$\nabla \cdot \vec{F} = 4z - 2y + y = 4z - y$$

$$\iiint_V (\nabla \cdot \vec{F})\, dv = \int_0^1\int_0^1\int_0^1 (4z-y)\, dx\, dy\, dz$$

$$= \int_0^1\int_0^1 (4xz - xy)_0^1 \, dy\, dz$$

dydz

$$= \int_0^1\int_0^1 [4z - y]\, dy\, dz$$

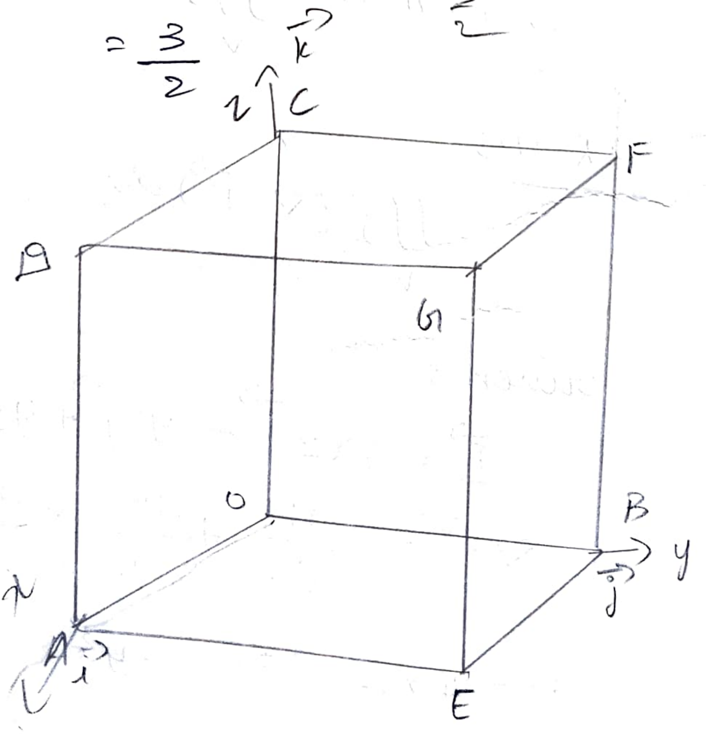
$$= \int_0^1 \left(4yz - \frac{y^2}{2}\right) dz$$

$$= \int_0^1 \left[4z - \frac{1}{2}\right] dx = \left[4\left(\frac{z^2}{2}\right) - \frac{z}{2}\right]_0^1$$

$$= 2z^2 - \frac{2}{2} = 2 - \frac{1}{2}$$

$\frac{3}{2}$

$$= \frac{3}{2}$$



S_1
S_2
S_3

	Face	$\hat{n}$	$\vec{F}\cdot\hat{n}$	Eqn	$\vec{F}\cdot\hat{n}$ on S	ds	$\iint_S(\vec{F}\cdot\vec{n})ds$
S_1	AEGD	$\vec{i}$	$4xz$	$x=1$	$4z$	$dydz$	$\iint 4z\,dydz$
S_2	OBFC	$-\vec{i}$	$-4xz$	$x=0$	0	$dydz$	0
S_3	EBFG	$\vec{j}$	$-y^2$	$y=1$	-1	$dxdz$	$\iint -1\,dxdz$
S_4	OCDA	$-\vec{j}$	$+y^2$	$y=0$	0	$dxdz$	0
S_5	GFCD	$\vec{k}$	yz	$z=1$	y	$dxdy$	$\iint y\,dxdy$
S_6	AEBO	$-\vec{k}$	$-yz$	$z=0$	0	$dxdy$	0.

μ.

$\vec{i}$

LoHoS :- LoHoS $\iint_S =$

$$\iint_S = \iint_{S_1} + \iint_{S_3} + \iint_{S_5}$$

$$\iint_{S_1} = \int_0^1\int_0^1 4z\,dydz$$

$$= \int_0^1\int_0^1 (4yz)_0^1\,dz$$

$$= \int_0^1 (4z)\,dz$$

$$= 4\left(\frac{z^2}{2}\right)_0^1$$

$$\iint_{S_1} = 2$$

$\Rightarrow \iint_{S_1} = \int_0^1\int_0^1 4z\,dydz$

$\Rightarrow 4z\,dydz$

$\Rightarrow \iint_{S_1} = \int_0^1\int_0^1 4z\,dydz$

$\Rightarrow 4z\,dydz$

$\int_0^1 (4z)\,dz$

$= 4\left(\frac{z^2}{2}\right)_0^1$

$= 2$.

$$\iint_{S_3} = \int_0^1\int_0^1 dx\,dz$$

$$= -\int_0^1 (x)_0^1\, dz$$

$$= -\int_0^1 dz$$

$$= -(z)_0^1$$

$$= -1.$$

$$\iint_{S_5} = \int_0^1\int_0^1 y\,dx\,dy$$

$$= \int_0^1 (xy)_0^1\, dy$$

$$= \int_0^1 y\,dy$$

$$= \left(\frac{y^2}{2}\right)_0^1$$

$$\iint_{S_5} = \frac{1}{2}.$$

<u>L.H.S</u>

$$\iint_S = \iint_{S_1} + \iint_{S_3} + \iint_{S_5}$$

$$= 2 - 1 + 1/2$$

$$= 3/2$$

$$L.H.S = R.H.S$$

Hence Gauss divergence theorem is Verified.

8) ~~Hence~~ Verify green's theorem

$\int (3x^2-8y^2)\,dx + (4y-6xy)\,dy$

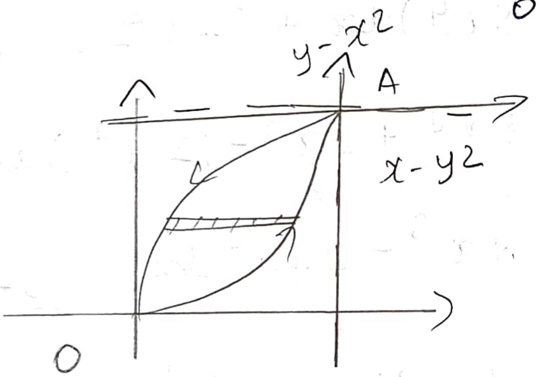
$x=y^2,\ y=x^2$

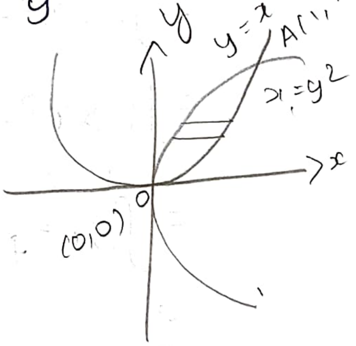
$\int u\,dx + v\,dy = \iint \left(\frac{dv}{dx} - \frac{du}{dy}\right) dx\,dy$

$u = 3x^2 - 8y^2,\ v = 4y - 6xy$

$\frac{du}{dy} = -16y,\ \frac{dv}{dx} = -6y$

$\iint\left(\frac{dv}{dx} - \frac{du}{dy}\right) dx\,dy = \int_0^1 \int_{y^2}^{\sqrt{y}}$





(in) $x = y^2$ to $x = \sqrt{y}$ $\boxed{x=\sqrt{y}}$

(out) $y=0$ to $y=1$ $dx\,dy$

$= \int_0^1 \int_{y^2}^{\sqrt{y}} (-6y + 16y)\, dx\,dy$

$= \int_0^1 \int_{y^2}^{\sqrt{y}} 10y\, dx\,dy = \int_0^1 \left[10xy\right]_{y^2}^{\sqrt{y}} dy$

$= \int_0^1 10y^{3/2} - 10y^3\, dy$

$= \left[10\,\frac{2}{5}\,y^{5/2} - 10\,\frac{y^4}{4^2}\right]_0^1$

$= 4 - \frac{5}{2} = \frac{8-5}{2} = 3/2$

$R.H.S = 3/2$

L.H.S

$\int u\,dx + v\,dy = \int_{OA} + \int_{AO}$

$\int_{OA} (3x^2 - 8y^2)\,dx + (4y - 6xy)\,dy$

$\left[y = x^2, \ \frac{dy}{dx} = 2x, \ dy = 2x\,dx \right]$

$= \int_0^1 (3x^2 - 8x^4)\,dx + (4x^2 - 6x^3)\,2x\,dx$

$= \int_0^1 (3x^2 - 8x^4 + 8x^3 - 12x^4)\,dx$

$= \int_0^1 3x^2 + 8x^3 - 24x^4\,dx$

$\left[x^3 + 2x^4 - \frac{24}{5} x^5 \right]_0^1$

$= \left[1 + 2 - \frac{24}{5} \right]$

$\int_{OA} = 1/5$

$\int_{AO} (3x^2 - 8y^2)\,dx + (4y - 6xy)\,dy$

$\left[x = y^2,\ y = \sqrt{x},\ \frac{dy}{dx} = \frac{1}{2}\frac{1}{\sqrt{x}},\right.$

$\left. dy = \frac{dx}{2\sqrt{x}} \right].$ $\qquad \frac{dx}{2\sqrt{x}}$

$= \int_1^0 (3x^2 - 8x)\,dx + (4\sqrt{x} - 6x^{3/2})\frac{dx}{2\sqrt{x}}$

$= \int_1^0 (3x^2 - 8x + 2 - \frac{6x}{2})\,dx$

$= \int_1^0 (3x^2 - 11x + 2)\,dx$

$= \left[x^3 - \frac{11}{2}x^2 + 2x\right]_1^0$

$= -1 + \frac{11}{2} - 2 = \frac{-6+11}{2}$

$\int_{AO} = \frac{5}{2}.$

$= \frac{5}{2} - 1$

$OA + AO = \frac{3}{2}.$

L.H.S = R.H.S

Hence green theorem verified

7) Verify gauss divergence theorem for $\vec{F} = (x^2 - yz)\vec{i} + (y^2 - 2x)\vec{j} + (z^2 - xy)\vec{k}$ over the rectangular Parallelopiped where $0 \le x \le a$, $0 \le y \le b$, $0 \le z \le c$.

by gauss divergence theorem

Ans:

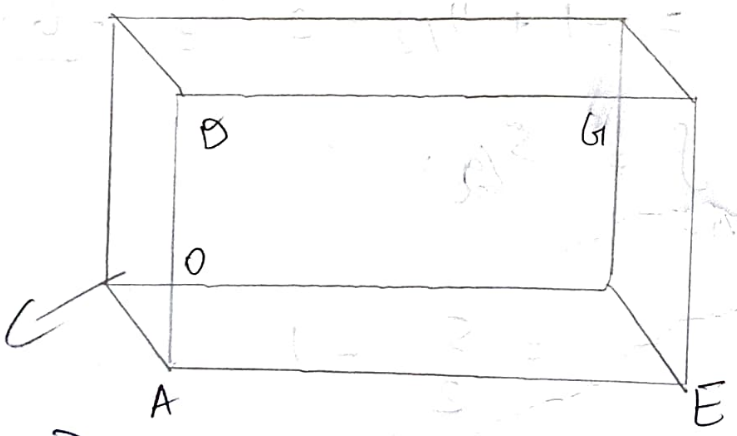
By GDT,

$$\iint_S \vec{F} \cdot \hat{n}\, ds = \iiint_V (\vec{\nabla} \cdot \vec{F})\, dv$$

Given:

$\vec{F} \cdot \hat{n}\, ds = \iiint_V (\nabla \cdot \vec{F})\, dv$

$$\vec{F} = (x^2 - yz)\vec{i} + (y^2 - 2x)\vec{j} + (z^2 - xy)\vec{k}$$



$$\nabla \cdot \vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

$F_1 = (x^2 - yz)$ $\quad F_2 = (y^2 - 2x)$

$F_3 = (z^2 - xy)$

$$\frac{\partial F_1}{\partial x} = 2x \quad \frac{\partial F_2}{\partial y} = 2y \quad \frac{\partial F_3}{\partial z} = 2z.$$

$$\nabla\cdot\vec{F} = 2x+2y+2z$$

$$=2(x+y+z).$$

$$\iiint_V (\nabla\cdot\vec{F})\,dv = \int_0^a\int_0^a\int_0^a 2(x+y+z)\,dx\,dy\,dz$$

R.H.S

$$= 2\int_0^a\int_0^b\int_0^a (x+y+z)\,dx\,dy\,dz$$

$$= 2\int_0^a\int_0^b\int_0^a \left(\frac{x^2}{2}+yx+zx\right)dy\,dz$$

$$=2\int_0^c\int_0^b\left[\frac{x^2}{2}+yx+zx\right]_0^a dy\,dz$$

$$=2\int_0^c\int_0^b\left[\frac{a^2}{2}+ay+az\right]dy\,dz$$

$$=2\int_0^c\left[\frac{a^2y}{2}+\frac{ay^2}{2}+azy\right]_0^b dz$$

$$=2\int_0^c\left[\frac{a^2b}{2}+\frac{ab^2}{2}+abz\right]dz$$

$$=2\left[\frac{a^2bz}{2}+\frac{ab^2z}{2}+\frac{abz^2}{2}\right]_0^c \cancel{dz}$$

$$=2\left[\frac{a^2bc}{2}+\frac{ab^2c}{2}+\frac{abc^2}{2}\right]$$

$$=\cancel{2}\left(\frac{abc}{\cancel{2}}\right)\left[a+b+c\right]$$

R.H.S $= a^2bc+ab^2c+abc^2 = abc[a+b+c]$

ds		Face	$\hat{n}$	$\vec{F}\cdot\hat{n}$	Eqn	$\vec{F}\cdot\hat{n}$ ds	$\iint_S (\vec{F}\cdot\hat{n}$
$dydz$	S_1	AEGD	$\vec{i}$	x^2-yz	$x=a$	a^2-yz	$\iint (a^2-yz$ $dydz$
$dydz$	S_2	OBFC	$-\vec{i}$	$yz-x^2$	$x=0$	yz	$\iint yz\,dy$
$dxdz$	S_3	EBFG	$\vec{j}$	y^2-2x	$y=b$	b^2-2x	$\iint b^2-2$ $dxdz$
$dxdz$	S_4	OCDA	$-\vec{j}$	$2x-y^2$	$y=0$	$2x$	$\iint 2x$ $dxdz$
$dxdy$	S_5	GFCD	$\vec{k}$	z^2-xy	$z=c$	c^2-xy	$\iint c^2-x$ $dxdy$
$dxdy$	S_6	AEBO	$-\vec{k}$	$xy-z^2$	$z=0$	xy	$\iint xy$ $dxdy$

L.H.S

$$S_1 = \int_0^c \int_0^b (a^2 - yz)\,dy\,dz$$

$$= \int_0^c \cancel{\tfrac{a^2}{2}} (a^2 - yz)\,dy\,dz$$

$$= \int_0^c \left(a^2y - \frac{y^2}{2} z\right)_0^b dz$$

$$= \int_0^c \left(a^2b - \frac{b^2}{2} z\right) dz$$

$$= \left[a^2bz - \frac{b^2}{2}\frac{z^2}{2}\right]_0^c$$

$$= a^2bc - \frac{b^2}{2}\frac{c^2}{2}$$

$$= a^2bc - \frac{b^2c^2}{4}.$$

$$S_2 = \int_0^c\int_0^b yz\,dy\,dz.$$

$$= \int_0^c \left(\frac{y^2z}{2}\right)_0^b dz$$

$$= \int_0^c \left(\frac{b^2}{2}z\right) dz$$

$$= \left[\frac{b^2}{2}\frac{z^2}{2}\right]_0^c$$

$$= \frac{b^2c^2}{4}.$$

$$S_3 = \int_0^c\int_0^a b^2 - zx\,dx\,dz$$

$$= \int_0^c \left[b^2x - \frac{zx^2}{2}\right]_0^a dz$$

$$= \int_0^c \left(b^2a - \frac{za^2}{2}\right) dz$$

$$= \left(b^2az - \frac{z^2a^2}{4}\right)_0^c$$

$$S_3 = b^2ac - \frac{c^2a^2}{4}$$

$$S_4 = \int_0^c\int_0^a 2x \, dx \, dz$$

$$= \int_0^c \left[2x^2 \right]_0^a dz$$

$$= \int_0^c \left[2a^2 \right] dz$$

$$= \left[\frac{z^2 a^2 \cancel{z}}{2} \right]_0^c$$

$$S_4 = \frac{\cancel{2}a^2 c^2}{4}.$$

$$S_5 = \int_0^b\int_0^a c^2 - xy \, dx \, dy$$

$$= \int_0^b \left[c^2 x - \frac{x^2}{2} y \right]_0^a dy$$

$$= \int_0^b \left[c^2 a - \frac{a^2}{2} y \right] dy$$

$$= \left[c^2 a y - \frac{a^2 y^2}{4} \right]_0^b$$

$$S_5 = c^2 ab - \frac{a^2 b^2}{4}$$

$$S_6 = \int_0^b\int_0^a xy \, dx \, dy$$

$$= \int_0^b \left[\frac{x^2 y}{2} \right]_0^a \cancel{dx}\, dy$$

$$= \int_0^b \left[\frac{a^2 y}{2}\right] dy$$

$$= \left[\frac{a^2 y^2}{4}\right]_0^b$$

$$= \frac{a^2 b^2}{4}$$

x+y=c

L.H.S

$$\iint_S = \iint_{S_1} + \iint_{S_2} + \iint_{S_3} + \iint_{S_4} + \iint_{S_5} + \iint_{S_6}$$

$$= \left[a^2 bc - \cancel{\frac{b^2 c^2}{4}}\right] + \left[\cancel{\frac{b^2 c^2}{4}}\right]$$

$$+ \left[b^2 ac - \cancel{\frac{c^2 a^2}{4}}\right] + \left[\cancel{\frac{a^2 c^2}{4}}\right]$$

$$+ \left[c^2 ab - \cancel{\frac{a^2 b^2}{4}}\right] + \left[\cancel{\frac{a^2 b^2}{4}}\right].$$

L.H.S = $abc\,[a+b+c]$. abc (a+c

R.H.S = L.H.S

Hence Gauss divergence theorem is verified.