Stoke's theorem:

Surface integral of the normal component of the curl of a vector function F over an open surface S = The line integral of the tangential component of F around the closed curve c boundings

$$\int_C \vec{F}\cdot d\vec{r} = \iint_S (\nabla\times\vec{F})\cdot\hat{n}\, ds.$$ *

1) Verify Stoke's theorem for $\vec{F} = (x^2 - y^2)\vec{i} + 2xy\vec{j}$ in the rectangular region in the xy plane bounded by the lines $x = 0$, $x = a$, $y = 0$, $y = b$.

Soln: By Stokes

By Stokes theorem.

$$\int_C \vec{F}\cdot d\vec{r} = \iint_S (\nabla\times\vec{F})\cdot\hat{n}\cdot ds$$

$$\int_C \vec{F}\cdot d\vec{r} = \iint_S (\nabla\times\vec{F})\cdot\hat{n}\, ds$$

Given:

$$\vec{F} = (x^2 - y^2)\vec{i} + 2xy\vec{j}$$

$$\nabla\times\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ (x^2-y^2) & 2xy & 0 \end{vmatrix}$$

$$= \vec{i}\left(\frac{\partial}{\partial y}(0) - \frac{\partial}{\partial x}(2xy)\right) - \vec{j}\left(\frac{\partial}{\partial x}(0) - \frac{\partial}{\partial z}(x^2-y^2)\right) + \vec{k}\left(\frac{\partial}{\partial x}(2xy) - \frac{\partial}{\partial y}(x^2-y^2)\right).$$

$$= 0\vec{i} - \vec{j}(0) + \vec{k}(2y+2y)$$

$$\boxed{\nabla \times \vec{F} = 4y\vec{k}} \qquad (\nabla \times \vec{F}) \cdot \vec{k} = (4y\vec{k}) \cdot \vec{k}$$

$\hat{k}$

$$= 4y.$$

heorem $\hat{n} = \vec{k}$ $\Rightarrow \hat{n} = \hat{k}$

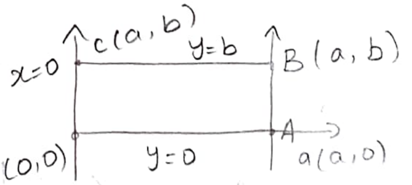
⇒ Stoke's theroem = Gauss divergence theorem

R.H.S

$$\iint_S (\nabla \times \vec{F})\,\hat{n}\,dy = \int_0^b \int_0^a 4y\,dx\,dy.$$

$$= \int_0^b (4xy)_0^a\,dy \qquad \iint = \int_0^b (4xy)_0^a\,dy$$

$$= \int_0^b (4ay)\,dy$$



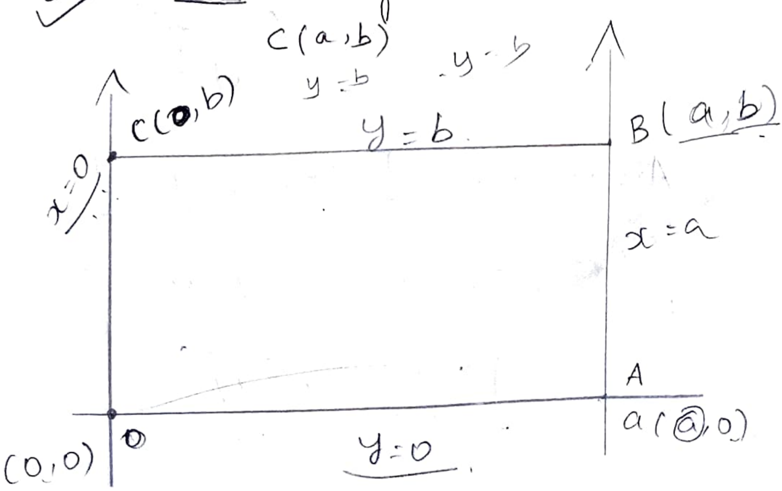
$$= \left(\frac{4ay^2}{2}\right)_0^b$$

⇒ $y=0$ ⇐

$$= (2ay^2)_0^b = 2ab^2 = 2ab^2$$

$$= (2ay^2)_0^b = 2ab^2$$

= R.H.S.



L.H.S

$$\int_C \vec{F}\cdot d\vec{r} = \int_{OA} + \int_{AB} + \int_{BC} + \int_{CA}$$

Given: Along

$$\vec{F} = (x^2 - y^2)\vec{i} + (2xy)\vec{j}$$

$$d\vec{r} = dx\,\vec{i} + dy\,\vec{j} + dz\,\vec{k}$$

$$\vec{F}\cdot d\vec{r} = (x^2 - y^2)\,dx + (2xy)\,dy$$

Along OA: $[y=0,\ dy=0]$

$y=0,\ dy=0$

$$\int_{OA} (x^2 - y^2)\,dx + 2xy\,dy =$$

$[y=0,\ dy=0]$ $\therefore$ $\int_0^a x^2\,dx$ $[y=0,\ dy=0]$

$$= \left(\frac{x^3}{3}\right)_0^a = \left(\frac{x^3}{3}\right)_0^a = \frac{a^3}{3}$$

Along AB: $[x=a,\ dx=0]$

$$\int_{AB} (x^2 - y^2)\,dx + 2xy\,dy = \int_0^b 2(a)y\,dy$$

$$= \left[2a\frac{y^2}{2}\right]_0^b$$

$$= ab^2$$

Along BC: $[y=b,\ dy=0]$

$$\int_{BC} (x^2 - y^2)\,dx + 2xy\,dy = \int_a^0 (x^2 - b^2)\,dx$$

$$= \left[\frac{x^3}{3} - b^2x\right]_a^0$$

$= -\frac{a^3}{3} + ab^2$

Along CO : $[x = 0, dx = 0]$.

$\int_{CO} (x^2 - y^2)\, dx + 2xy\, dy = 0$

Along CO : Along BC : Along AB : Along CO :

L.H.S

$\int \vec{f} \cdot d\vec{r} = \int_{OA} + \int_{AB} + \int_{BC} + \int_{CO}$

L.H.S $= \frac{a^3}{3} + ab^2 - \frac{a^3}{3} + ab^2 + 0$

$\Rightarrow = 2ab^2$ $= 2ab^2 = $ L.H.S

$= 2ab^2$
$=$ L.H.S

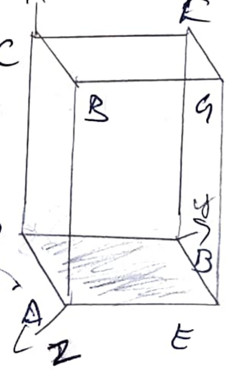
$\Rightarrow =$ L.H.S R.H.S $=$ L.H.S

$\Rightarrow$ L.H.S $=$ R.H.S $\Rightarrow$ Stokes theorem

Hence Stokes theorem is verified $\rightarrow$ verified $\leftarrow$ theorem Stoke's

Verify Stoke's theorem for

2) verify Stoke's theorem for $\vec{F} = (y - z + 2)\vec{i} + (yz + 4)\vec{j} - xz\vec{k}$ over the open surfaces of the cube, $x = 0, y = 0, z = 0$, $x = 1, y = 1, z = 1$.



Sln:

By Stokes theorem :

$\int_C \vec{F} \cdot d\vec{r} = \iint_S (\nabla \times \vec{F}) \cdot \hat{n}\, ds$

$\vec{F} \cdot d\vec{r}$

Given :

$\vec{F} = (y - z + 2)\vec{i} + (yz + 4)\vec{j} - xz\vec{k}$.

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ (y-z+2) & (yz+4) & -xz \end{vmatrix}$$

$(\nabla \times \vec{F})\cdot\hat{n}$

$(\nabla \times \vec{F})\cdot\hat{n}$

$$= \vec{i}\left(\frac{\partial}{\partial y}(-xz) - \frac{\partial}{\partial z}(yz+4)\right)$$

$$-\vec{j}\left(\frac{\partial}{\partial x}(-xz) - \frac{\partial}{\partial z}(+y-z+2)\right)$$

$$+\vec{k}\left(\frac{\partial}{\partial x}(y-z+2) - \frac{\partial}{\partial y}(y-z+2)\right).$$

$$= \vec{i}(0 - y) - \vec{j}(-z+1) + \vec{k}(0 - 1).$$

$$= -y\vec{i} + (z-1)\vec{j} - \vec{k}.$$

$\vec{F}\cdot\hat{n}$	faces	$\hat{n}$	$(\nabla \times \vec{F})\cdot\hat{n}$	Eqn	ds	$\iint \vec{F}\cdot\hat{n}\,ds$
$-y$	S_1 AEGD	$\vec{i}$	$-y$	$x=1$	$dy\,dz$	$\iint(-y)\,dy\,dz$
y	S_2 OBFC	$-\vec{i}$	y	$x=0$	$dy\,dz$	$\iint(y)\,dy\,dz$
$z+1$	S_3 EBFG	$\vec{j}$	$z+1$	$y=1$	$dx\,dz$	$\iint(z+1)\,dx\,dz$
$-z-1$	S_4 AOCO	$-\vec{j}$	$-z-1$	$y=0$	$dx\,dz$	$\iint(-z-1)\,dx\,dz$
-1	S_5 CDGE	k	-1	$z=1.$	$dx\,dy$	$\iint(-1)\,dx\,dy$

R.H.S.

Q) $\iint_S \nabla\times\vec{F}\cdot\hat{n}\,ds = \iint_{S_1} + \iint_{S_2} + \iint_{S_3} + \iint_{S_4} + \iint_{S_5}$

i) $\iint_{S_1} \nabla\times\vec{F}\cdot\hat{n}\,ds = \iint (-y\vec{i} + (z-1)\vec{j} + \vec{k}) \cdot \vec{i}\,dy\,dz$

$x=1$

$= \int_0^1\int_0^1 -y\,dy\,dz = -1/2$

ii) $\iint_{S_2} \nabla\times\vec{F}\cdot\hat{n}\,ds = \iint(-y\vec{i} + (z-1)\vec{j} - \vec{k}) \cdot (-\vec{i})\,dy\,dz$

$\iint_{S_2}(\nabla\times\vec{F}\cdot\hat{n})\,ds.\ \iint(-y\vec{i}$

$= \int_0^1\int_0^1 y\,dy\,dz = 1/2$

iii) $\iint_{S_3} \nabla\times\vec{F}\cdot\hat{n}\,ds = \iint(-y\vec{i} + (z-1)\vec{j} - \vec{k}) \cdot \vec{j}\,dx\,dz$

$= \int_0^1\int_0^1 (z-1)\,dx\,dz = -1/2$

iv) $\iint_{S_4} \nabla\times\vec{F}\cdot\hat{n}\,ds = \iint(-y\vec{i} + (z-1)\vec{j} - \vec{k}) \cdot (-\vec{j})\,dx\,dz$

$= \int_0^1\int_0^1 -(z-1)\,dx\,dz = 1/2$

v) $\iint_{S_5} \nabla\times\vec{F}\cdot\hat{n}\,ds = \iint(-y\vec{i} + (z-1)\vec{j} - \vec{k}) \cdot \vec{k}\,dx\,dy$

$= \int_0^1\int_0^1 (-1)\,dx\,dy = -1.$

R.H.S $= -1/2 + 1/2 - 1/2 + 1/2 - 1$

$= -1.$ $\quad \frac{-1}{2}$

$\vec{F} = (y - z + 2)\vec{i} + (yz+4)\vec{j} - xz\vec{k}$

$d\vec{r} = dx\vec{i} + dy\vec{j} + dz\vec{k}$

$\vec{F} \cdot d\vec{r} = (y - z + 2)\,dx + (yz+4)\,dy - xz\,dz.$

<u>L.H.S</u>

$$\int_C \vec{F} \cdot d\vec{r} = \int_{OA} + \int_{AE} + \int_{EB} + \int_{BO}$$

i) $\displaystyle\int_{OA} \vec{F} \cdot d\vec{r} = \int_{OA} (y-z+2)\,dx + (yz+4)\,dy - xz\,dz$

Along OA, x varies from 0 to 1

$y = 0,\ dy = 0$

$z = 0,\ dz = 0$

$$= \int_0^1 2\,dx = 2.$$

ii) $\displaystyle\int_{AE} \vec{F} \cdot d\vec{r} = \int_{AE} (y-z+2)\,dx + (yz+4)\,dy - xz\,dz$

Along AE, y varies from 0 to 1

$x = 0,\ dx = 0$

$z = 0,\ dz = 0$

$$= \int_0^1 4\,dy = 4$$

iii) $\int_{EBC} \vec{F}\cdot d\vec{r} = \int\int_{EBC} (y-z+2)dx + (yz+4)dy - xzdz$

x varies from 1 to 0

$y = 0$, $dy = 0$

$z = 0$, $dz = 0$.

$= \int_1^0 3dx = -3$.

iv) $\int_{BO} \vec{F}\cdot d\vec{r} = \int\int_{BO} (y-z+2)dx + (yz+4)dy - xzdz$

y varies from 1 to 0

$x = 1$ $x = 0$, $dx = 0$

$x = 0$ to 1
$y = 0$ to 0

$z = 0$, $dz = 0$

$= \int_1^0 4dy = -4$.

$\int_{OA} + \int_{AE} + \int_{EBC} + \int_{BO} = 2 + 4 - 3 - 4$

$= -1$.

L.H.S = R.H.S

Hence Stoke's theorem is verified.