23MAT102 – Complex Analysis and Laplace Transforms

IAE – III Ans Key

Part – A

1. Find $L\left[\frac{1}{\sqrt{t}}\right]$

soln

$$L[t^n] = \frac{\sqrt{n+1}}{s^{n+1}}$$

$$L\left[\frac{1}{\sqrt{t}}\right] = L\left[t^{-1/2}\right] = \frac{\sqrt{1/2}}{s^{1/2}} = \sqrt{\frac{\pi}{s}}$$

2. First Shifting theorem

* If $L[f(t)] = F(s)$, then $L[e^{at} f(t)] = f(s-a)$

* If $L[f(t)] = [F(s)]$, then $L[e^{-at} f(t)] = F(s+a)$

Proof: $L[e^{at} f(t)] = \int e^{-st} e^{at} f(t)dt =$

$$= \int_0^\infty e^{-(s-a)t} e^{at} f(t)\,dt = \int_0^\infty e^{-(s-a)t} f(t)dt = F(s-a)$$

$$L[e^{-at} f(t)] = \int_0^\infty e^{-st} e^{-at} f(t)\,dt = \int_0^\infty e^{(s+a)t} f(t)dt = F(s+a)$$

3. Laplace Transforms

Initial value Theorem

$$\lim_{t\to 0} f(t) = \lim_{s\to\infty} sF(s)$$

Final value Theorem

$$\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)$$

4. Evaluate $L^{-1}\left[\frac{1}{s^2-6s+5}\right]$

soln $= s^2 - 6s + 5 \Rightarrow (s-3)^2 - 4$

$$L^{-1}\frac{1}{(s+a)^2+b^2} = \frac{1}{b} e^{-at} \sin bt$$

$a = 3, \quad b = 2$

$$L^{-1}\frac{1}{(s+3)^2+2^2} = \frac{1}{2} e^{-3t} \sin 2t$$

5. inverse Laplace of $\frac{s}{(s+2)^2}$

soln $= L^{-1}\left[\frac{s}{(s+2)^2}\right]$

$$= \frac{d}{dt} L^{-1}\left[\frac{1}{(s+2)^2}\right] = \frac{d}{dt}\left(e^{-2t} L^{-1}\left[\frac{1}{s^2}\right]\right)$$

$$= \frac{d}{dt}\left[e^{-2t}(t)\right] = e^{-2t}(1) - 2 + e^{-2t}$$

$$= e^{-2t}[1-2t]$$

Part – B

6. a) Find $L\left[\frac{\cos at - \cos bt}{t}\right]$

soln

$$L\left[\frac{\cos at - \cos bt}{t}\right] = \int_0^\infty L(\cos at - \cos bt)\,ds$$

$$= \int_s^\infty [L(\cos t) - L(\cos bt)]\,ds$$

$$= \int_s^\infty \left[\frac{s}{(s^2+a^2)} - \frac{s}{(s^2+b^2)}\right] ds$$

$$= \frac{1}{2}\int_s^\infty \left(\frac{2s}{s^2+a^2} - \frac{2s}{s^2+b^2}\right) ds$$

$$= \frac{1}{2}\left[\log(s^2+a^2) - \log(s^2+b^2)\right]_s^\infty$$

$$= \frac{1}{2}\left[\log\left(\frac{s^2+a^2}{s^2+b^2}\right)\right]_s^\infty$$

$$= \frac{1}{2}\left[\log\frac{s^2(1+a^2/s^2)}{s^2(1+b^2/s^2)}\right]_s^\infty$$

$$= \frac{1}{2}\left[\log 1 - \log(s^2+a^2/s^2+b^2)\right]$$

$$= \frac{1}{2}\left[0 - \log(s^2+a^2/s^2+b^2)\right]$$

$$= \frac{1}{2}\left[\log(s^2+a^2/s^2+b^2)^{-1}\right]$$

$$= \frac{1}{2}\left[\log(s^2+b^2/s^2+a^2)\right]$$

b) Laplace Transform prove that $\int_0^\infty \frac{1-\cos 2t}{t^2} dt = \pi$

Soln $\int_0^\infty \frac{1-\cos 2t}{t^2} dt = L\left[\frac{1-\cos 2t}{t^2}\right]_{s=0}$

$L\left[\frac{1-\cos 2t}{t^2}\right] = \int_s^\infty \int_s^\infty L[1-\cos 2t] ds\, ds$

$= \int_s^\infty \int_s^\infty \left(\frac{1}{s} - \frac{s}{s^2+4}\right) ds\, ds$

$= \int_s^\infty \left[\log s - \frac{1}{2}\log(s^2+4)\right]_s^\infty ds$

$= \int_s^\infty \left[\log \frac{s}{\sqrt{s^2+4}}\right]_s^\infty ds = \int_s^\infty \left(0 - \log\frac{s}{\sqrt{s^2+4}}\right) ds$

$= \int_s^\infty \left[\log \frac{\sqrt{s^2+4}}{s}\right] ds = \frac{1}{2}\int_0^\infty \log \frac{s^2+4}{s^2} ds$

$= \frac{1}{2}\int_s^\infty \log(1+4/s^2)\, ds$

$= \left[\frac{1}{2} s \log(1+4/s^2)\right]_s^\infty$

$= \frac{1}{2}\int_s^\infty s \frac{1}{1+4/s^2}(-8/s^3) ds$

$= 0 - \frac{1}{2} s \log\left(\frac{s^2+4}{s^2}\right) + \frac{8}{2}\int_s^\infty \frac{1}{s^2+4} ds$

$= 0 + 4\int_s^\infty \frac{1}{s^2+4} ds = \left[4 \cdot \frac{1}{2}\tan^{-1}\left(\frac{s}{2}\right)\right]_s^\infty$

$= \left[2\tan^{-1}(s/2)\right]_s^\infty = 2\frac{\pi}{2} - 2\tan^{-1} s/2$

$= \pi - 2\tan^{-1}(s/2)$

$\int_0^\infty \frac{1-\cos 2t}{t^2} dt =$

$[\pi - 2\tan^{-1} s/2]_{s=0}$

$= [\pi - 0] = \pi$

7. a) Using Convolution th...

i) $L^{-1}\left[\frac{1}{(s+a)(s+b)}\right]$:

Soln $L^{-1}\left[\left(\frac{1}{s+a}\right)\left(\frac{1}{s+b}\right)\right]$

$= L^{-1}\left(\frac{1}{s+a}\right) * L^{-1}\left(\frac{1}{s+b}\right)$

$= \int_0^t e^{-au} e^{-b(t-u)} du$

$= \int_0^t e^{-au} e^{-bt} e^{bu} du$

$= e^{-bt}\int_0^t e^{-(a-b)u} du$

$= e^{-bt}\left[\frac{e^{(-a-b)u}}{-(a-b)}\right]_0^t$

$= \frac{e^{-bt}}{-(a-b)}\left(e^{-(a-b)t} - 1\right)$

$= \frac{e^{-at+bt-bt} - e^{-bt}}{-(a-b)}$

$= \frac{e^{-at} - e^{-bt}}{-(a-b)}$

ii) $L^{-1}\left[\frac{s}{(s^2+a^2)^2}\right]$

soln $L^{-1}\left[\frac{s}{s^2+a^2} \cdot \frac{1}{s^2+a^2}\right]$

$= L^{-1}\left[\frac{s}{s^2+a^2}\right] * L^{-1}\left[\frac{1}{s^2+a^2}\right]$

$= \cos at * \frac{1}{a}\sin at$

$= \frac{1}{a}\int_0^t [\cos au \cdot \sin a(t-u)] du$

$= \frac{1}{a}\int_0^t [\cos au \sin(at-au)] du$

$au = A$; $at - au = B$

$\therefore \cos A \sin B = \frac{1}{2}[\sin(A+B) + \sin(A-B)]$

$= \frac{1}{2}\left[\int_0^t \sin(au+at-au) + \sin(au-at+au)\right] du$

$= \frac{1}{2}a\left[\int_0^t \sin at + \sin(2au - at)\right]du$

$= \frac{1}{2}a\left[(\sin at)u - \frac{\cos 2au - at}{2a}\right]_0^t$

$= \frac{1}{2a}(\sin at)t - \frac{\cos(2at - at)}{2a} - \frac{\cos 2a(0) - at}{2a}\Big]$

$= \frac{1}{2a}\left[t\sin at - \frac{\cos at}{2a} + \frac{\cos at}{2a}\right]$

$= \frac{1}{2a}[t \sin at]$

b) Using Partial fraction, find inverse Laplace transform $\left[\frac{2}{(s+1)(s^2+4)}\right]$

Soln

$= L^{-1}\left[\frac{1}{s+1} \cdot \frac{2}{s^2+4}\right]$

$= L^{-1}\left[\frac{1}{s+1}\right] * L^{-1}\left[\frac{2}{s^2+4}\right]$

$= e^{-t} * \sin 2t$

$= \sin 2t * (e^{-t})$

$= \int_0^t \sin 2t * (e^{-t})$

$= \int_0^t \sin 2u\, e^{-(t-u)} du$

$= \int_0^t \sin 2u\, e^{-t} e^{u} du$

$= e^{-t}\int_0^t e^{u} \sin 2u\, du$

$= e^{-t}\left[\frac{e^u}{1^2+2^2}(\sin 2u - 2\cos 2u)\right]_0^t$

$= e^{-t}\left[\frac{e^t}{5}(\sin 2t - 2\cos 2t) - \frac{1}{5}(0-2)\right]$

$= \frac{1}{5}(\sin 2t - 2\cos 2t) + \frac{2}{5}e^{-t}$

8. a) $f(t) = \begin{cases} t, & 0 \le t \le a \\ 2a - t, & \text{if } a \le t \le 2a \end{cases}$ and $f(t+2a) = f(t)$

Soln

$L[f(t)] = \frac{1}{1-e^{-ps}}\int_0^p e^{-st} f(t)\,dt$

$P = 2a$

$L[f(t)] = \frac{1}{1-e^{-2as}}\int_0^{2a} e^{-st} f(t)\,dt$

$= \frac{1}{1-e^{-2as}}\left\{\int_0^a e^{-st}(t)\,dt + \int_a^{2a} e^{-st}(2a-t)\,dt\right\}$

$u = t \qquad \int dv = \int e^{-st}dt$

$u' = 1 \qquad v = \frac{-e^{-st}}{s}$

$u'' = 0 \qquad v_1 = -\frac{1}{s}\left[\frac{e^{-st}}{-s}\right]$

$u = 2a - t \qquad \int dv = \int e^{-st} dt$

$u' = -1 \qquad v = \frac{e^{-st}}{-s}$

$u'' = 0 \qquad v_1 = \frac{-1}{s}\left(\frac{e^{-st}}{-s}\right)$

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$\int u\,dv = uv - u'v_1 + u''v_2$

$L[f(t)] = \frac{1}{1-e^{-2as}}\left[t\left(\frac{e^{-st}}{s}\right) - 1\left(\frac{1}{s^2}e^{-st}\right)\right]_0^a + \left[(2a-t)\left(\frac{e^{-st}}{-s}\right) + 1\left(\frac{e^{-st}}{s^2}\right)\right]_a^{2a}$

$= \frac{1}{1-(e^{-as})^2}\left[\frac{ae^{-sa}}{s} - \frac{e^{-sa}}{s^2} - 0 + \frac{1}{s^2} + 0 + \frac{e^{-2as}}{s^2} + \frac{ae^{-sa}}{s} - \frac{e^{-sa}}{s^2}\right]$

$= \frac{1}{1-(e^{-as})^2}\left[\frac{-2a^{-sa}}{s^2} + \frac{1}{s^2} + \frac{e^{-2as}}{s^2}\right]$

$= \frac{1}{(1+e^{-as})(1-e^{-as})}\frac{1}{s^2}\left[1 + (e^{-as})^2 - 2e^{-as}\right]$

$= \frac{1}{s^2(1+e^{-as})(1-e^{-as})}\left[(1-e^{-as})^2\right]$

$= \frac{1-e^{-as}}{\ldots}$

b) Solve $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 2y = 0$,

$y = \frac{dy}{dx} = 1$ at $x = 0$ using

Laplace transform method.

Soln

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 2y = 0$$

Initial conditions

$y(0) = 1 \quad y'(0) = 1$

Let $y(s) = L[y(x)]$

$$L\left[\frac{d^2y}{dx^2}\right] = L[y(x)]$$

$$L\left(\frac{d^2y}{dx^2}\right) = s^2 y(s) - sy(0) - y(0)$$

$$L\left(\frac{dy}{dx}\right) = sy(s) - y(0)$$

$$L(y) = y(s)$$

$$s^2 y(s) - s(1) - 1 - 2(sy(s) - 1) + 2y(s) = 0$$

$$(s^2 y(s) - s - 1 - 2sy(s) + 2y(s)) = 0$$

$$(s^2 - 2s + 2)\, y(s) = s - 1$$

$$y(s) = \frac{s-1}{s^2 - 2s + 2}$$

$$y(s) = \frac{s-1}{(s-1)^2 + 1}$$

$$L[e^{ax}\cos(bx)] = \frac{s}{s^2 + a^2}$$

$$L[e^{ax}\sin(bx)] = \frac{b}{s^2 + b^2}$$

$$L^{-1}(y(s)) = L^{-1}\left(\frac{s-1}{(s-1)^2 + 1}\right)$$

$$= e^{x}\cos(x)$$

$$y(x) = e^{x}\cos x$$