



TOPIC : 6 - HALF RANGE SINE SERIES

Half range sine series in the interval $(0, \pi)$ $(0, l)$

Formula : $(0, \pi)$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

$$\text{where } b_n = \frac{2}{b-a} \int_a^b f(x) \sin nx dx$$

Formula: $(0, l)$

$$f(x) = \sum_{n=1}^{\infty} b_n \frac{\sin n\pi x}{l}$$

$$\text{where } b_n = \frac{2}{b-a} \int_a^b f(x) \frac{\sin n\pi x}{l} dx$$

Problems:

① $f(x) = x(\pi-x)$ in $0 < x < \pi$.

Sol:

$$f(x) = x(\pi-x)$$
$$= x\pi - x^2$$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

$$b_n = \frac{2}{b-a} \int_a^b f(x) \sin nx dx$$

$$u = x\pi - x^2 \quad dv = \sin nx$$

$$u_1 = \pi - 2x \quad v = -\frac{\cos nx}{n}$$

$$u_2 = -2 \quad v_1 = -\frac{\sin nx}{n^2}$$

$$v_2 = \frac{\cos nx}{n^3}$$

$$= \frac{2}{\pi} \int_0^{\pi} (x\pi - x^2) \sin nx dx$$

$$= \frac{2}{\pi} \left[(x\pi - x^2) \frac{\cos nx}{n} + (\pi - 2x) \frac{\sin nx}{n^2} \right. \\ \left. - 2 \frac{\cos nx}{n^3} \right]_0^{\pi}$$



$$b_n = \frac{4}{\pi n^3} [1 - (-1)^n]$$

$$= \begin{cases} \frac{8}{\pi n^3} & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases}$$

$$f(x) = \sum_{n=1}^{\infty} \frac{8}{\pi n^3} \sin nx$$

$$f(x) = \frac{8}{\pi} \sum_{n=1}^{\infty} \frac{1}{n^3} \sin nx$$

Put $x = \frac{\pi}{2}$

$$f\left(\frac{\pi}{2}\right) = \frac{8}{\pi} \sum_{n=1}^{\infty} \frac{\sin n\left(\frac{\pi}{2}\right)}{(2n-1)^3}$$

$$\frac{\pi}{2} \left(\frac{\pi}{2}\right) = \frac{8}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n-1)^3}$$

$$\frac{\pi}{2} \cdot \frac{\pi}{2} \cdot \frac{\pi}{8} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n-1)^3}$$

$$\frac{\pi^3}{8} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n-1)^3}$$

① Obtain the sine series for the function

$$f(x) = \begin{cases} x, & 0 \leq x \leq \frac{l}{2} \\ l-x, & \frac{l}{2} \leq x \leq l \end{cases}$$

Sol:

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

$$b_n = \frac{2}{l} \int_0^l f(x) \sin n\pi x/l dx$$



$$\begin{array}{l} u = x \quad dv = \frac{\sin n\pi x}{x} \\ u_1 = 1 \quad v = -\frac{\cos n\pi x}{\frac{n\pi}{x}} \\ u_2 = 0 \quad v_1 = \frac{-\sin n\pi x}{(\frac{n\pi}{x})^2} \end{array} \quad \begin{array}{l} u = l-x \\ u_1 = -1 \\ u_2 = 0 \end{array}$$

$$b_n = \frac{2}{l} \left[\left(-x \frac{\cos n\pi x}{\frac{n\pi}{x}} + \frac{\sin n\pi x}{(\frac{n\pi}{x})^2} \right) \Big|_0^{l/2} + \left(-(l-x) \frac{\cos n\pi x}{\frac{n\pi}{x}} - \frac{\sin n\pi x}{(\frac{n\pi}{x})^2} \right) \Big|_{l/2}^l \right]$$

$$b_n = \frac{2}{l} \left[-\frac{l}{2} \frac{1}{\frac{n\pi}{l}} + \left(\frac{l}{2} \right) \frac{1}{\frac{n\pi}{l}} \right]$$

$$b_n = \frac{2}{l} \left[-\frac{l^2}{2} \frac{\cos n\pi}{n\pi} + \frac{l^2}{n^2\pi^2} \sin \frac{n\pi}{2} + \frac{l^2 \sin n\pi}{n^2\pi^2} + \frac{l^2}{2n\pi} \cos \frac{n\pi}{2} + \frac{l^2}{n^2\pi^2} \sin \frac{n\pi}{2} \right]$$

$$= \frac{2}{l} \left(\frac{2l^2}{n^2\pi^2} \sin \frac{n\pi}{2} \right)$$

$$b_n = \frac{4l}{n^2\pi^2} \sin \frac{n\pi}{2}$$

$$f(x) = \sum_{n=1}^{\infty} \frac{4l}{n^2\pi^2} \sin \frac{n\pi}{2} \frac{\sin n\pi x}{l}$$



③ $f(x) = (2-x)$ in $(0, 2)$.

Sol:

$$b_n = \frac{2}{2} \int_0^2 (2-x) \sin \frac{n\pi x}{2} dx$$

$$u = 2-x \quad dv = \sin \frac{n\pi x}{2}$$

$$v = -\frac{\cos \frac{n\pi x}{2}}{\frac{n\pi}{2}}$$

$$u_1 = -1 \quad v_1 = -\frac{\sin \frac{n\pi x}{2}}{\left(\frac{n\pi}{2}\right)^2}$$

$$b_n = \left[-(2-x) \frac{\cos \frac{n\pi x}{2}}{\frac{n\pi}{2}} - \frac{\sin \frac{n\pi x}{2}}{\left(\frac{n\pi}{2}\right)^2} \right]_0^2$$

$$= 2 \frac{\cos n\pi}{n\pi}$$

$$= \frac{4}{n\pi}$$

$$f(x) = \sum_{n=1}^{\infty} \frac{4}{n\pi} \sin \frac{n\pi x}{2}$$



$$a_n = \frac{2}{\pi} \int_0^{\pi} x \cos nx \, dx$$

$$= \frac{2}{\pi} \left[x \frac{\sin nx}{n} + \frac{\cos nx}{n^2} \right]_0^{\pi}$$

$$= \frac{2}{\pi} \left[\frac{(-1)^n}{n^2} - \frac{1}{n^2} \right]$$

$$a_n = \frac{2}{n^2 \pi} [-1 + (-1)^n]$$

$$a_n = \begin{cases} \frac{-4}{n^2 \pi} & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases}$$

$$f(x) = \frac{\pi}{2} + \sum_{n=1}^{\infty} \frac{2}{n^2 \pi} [1 - (-1)^n] \cos nx$$

$$f(x) = \frac{\pi}{2} + \sum_{n=\text{odd}} \frac{-4}{n^2 \pi} \cos nx$$

2. Find the half range cosine series of the function $f(x) = x(\pi - x)$ in $(0, \pi)$.

Sol:

$$f(x) = x\pi - x^2$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

$$a_0 = \frac{2}{\pi} \int_0^{\pi} (x\pi - x^2) \, dx$$

$$= \frac{2}{\pi} \left[\frac{x^2 \pi}{2} - \frac{x^3}{3} \right]_0^{\pi}$$

$$= \frac{2}{\pi} \left[\frac{\pi^3}{3} - \frac{\pi^3}{3} \right]$$