TOPIC : 1 –FOURIER INTEGRAL

(1)

Fourier Transform.

Complex form of F

Fourier Integral theorem:

If $f(x)$ is piecewise continuously differentiable and absolutely integrable in $(-\infty, \infty)$ then

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(t)\, e^{i(x-t)s}\, dt\, ds.$$

Fourier Cosine Integral formula:

$$f(x) = \frac{2}{\pi} \int_0^{\infty} \int_0^{\infty} f(t) \cos xs \cos st\, dt\, ds$$

which is called the F.C.I. of $f(x)$

Fourier Sine Integral formula.

$$f(x) = \frac{2}{\pi} \int_0^{\infty} \int_0^{\infty} f(t) \sin xs \sin st\, dt\, ds.$$ (2)

Example 1:

A function $f(x)$ is defined by

$$f(x) = \begin{cases} 1, & |x| \le 1 \\ 0, & |x| > 1 \end{cases}$$

Find the Fourier Integ representation of $f(x)$. Hence Evaluate

$$\int_0^{\infty} \frac{\sin s}{s} \cos sx\, ds$$

Fourier Integral Representation:

$$f(x) = \frac{1}{\pi}\left[\int_0^\infty A(s)\cos(xs) + B(s)\sin(xs)\right] ds$$

where $A(s) = \int_{-\infty}^{\infty} f(t)\cos st\,dt$ & $B(s) = \int_{-\infty}^{\infty} f(t)\sin st\,dt$.

Soln:

The Fourier integral representation of $f(x)$ is given by.

$$f(x) = \frac{1}{\pi}\int_0^\infty [A(s)\cos xs + B(s)\sin xs]\,ds$$

$$A(s) = \int_{-\infty}^{\infty} f(t)\cos st\,dt \text{ and } B(s) = \int_{-\infty}^{\infty} f(t)\sin st\,dt$$

$$A(s) = \int_{-1}^{1}\cos st\,dt = 2\int_0^1 \cos st\,dt = 2\left[\frac{\sin st}{s}\right]_0^1$$

$$= \frac{2\sin s}{s} \qquad \text{(even)}$$

$$B(s) = \int_{-1}^{1}\sin st\,dt = 0 \qquad [\because \sin st \text{ is an odd fun.}$$

$$\therefore f(x) = \frac{2}{\pi}\int_0^\infty \frac{\sin s}{s}\cos xs\,ds \longrightarrow (1)$$

This is the Fourier integral representation of $f(x)$.

(1) $$\frac{2}{\pi}\int_0^\infty \frac{\sin s}{s}\cos xs\,dx = f(x) = \begin{cases} 1, & |x| \le 1 \\ 0, & |x| > 1 \end{cases}$$

$$\therefore \int_0^\infty \frac{\sin s}{s}\cos xs\,ds = \begin{cases} \frac{\pi}{2}, & |x| \le 1 \\ 0, & |x| > 1 \end{cases}$$

Example 2:

A function $f(x)$ is defined as $f(x) = \begin{cases} 1, & 0<x<1 \\ 0, & x>1 \end{cases}$ S.T. in $0<x<1$, using Fourier Cosine integral,

$$\int_0^\infty \frac{\sin s}{s} \cos xs \, ds = \frac{\pi}{2}.$$

Soln:

The F.C.I. of $f(x)$ is given by

$$f(x) = \frac{2}{\pi} \int_0^\infty \int_0^\infty f(t) \cos st \cos xs \, dt \, ds$$

$$= \frac{2}{\pi} \int_0^\infty A(s) \cos xs \, ds \quad \text{where } A(s) = \int_0^\infty f(t) \cos st \, dt$$

$$= \int_0^1 1 . \cos st \, dt$$

$$= \left[\frac{\sin st}{s} \right]_0^1$$

from ①,

$$A(s) = \frac{\sin s}{s} \rightarrow ①$$

from ①,

$$f(x) = \frac{2}{\pi} \int_0^\infty \frac{\sin s}{s} \cos xs \, ds.$$

In $0<x<1$, $f(x) = 1$.

$$\frac{2}{\pi} \int_0^\infty \left(\frac{\sin s}{s}\right) \cos xs \, ds = 1.$$

$$\Rightarrow \int_0^\infty \frac{\sin s}{s} \cos xs \, ds = \frac{\pi}{2}.$$