TOPIC: 5 - **Fourier cosine Transforms**

Find F.C.T. of $f(x) = \begin{cases} x, & 0 < x < 1 \\ 2-x, & 1 < x < 2 \\ 0, & x > 2. \end{cases}$

Soln:

$$F_c\{f(x)\} = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos sx \, dx.$$

$$= \sqrt{\frac{2}{\pi}} \left\{ \int_0^1 x \cos sx \, dx + \int_1^2 (2-x) \cos sx \, dx \right\}$$

$$= \sqrt{\frac{2}{\pi}} \left\{ \left(x \frac{\sin sx}{s} + (1) \frac{\cos sx}{s^2} \right)_0^1 + \left((2-x) \frac{\sin sx}{s} - 1 \frac{\cos sx}{s^2} \right)_1^2 \right\}$$

$$= \sqrt{\frac{2}{\pi}} \left\{ \left(\cancel{\frac{\sin s}{s}} + \frac{\cos s}{s^2} - \frac{1}{s^2} \right) + \left(0 - \frac{\cos 2s}{s^2} - \cancel{\frac{\sin s}{s}} + \frac{\cos s}{s^2} \right) \right\}$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{2 \cos s}{c^2} - \frac{\cos 2s}{c^2} - \frac{1}{s^2} \right]$$

Example 16:
Find F.C.T. of $e^{-a^2x^2}$ and hence evaluate F.S.T. of $x e^{-a^2x^2}$.

Soln:

$$F_c\left(e^{-a^2x^2}\right) = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-a^2x^2} \cos sx \, dx.$$

$$= R.P. \text{ of } \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-a^2x^2} \cdot e^{isx} \, dx$$

$$= R.P. \text{ of } \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-a^2x^2 + isx} \, dx.$$

$$= R.P. \text{ of } \frac{1}{a\sqrt{2}} \cdot e^{-\frac{s^2}{4a^2}} \quad (eg: 4).$$

$$F_s\left(x e^{-a^2x^2}\right) = -\frac{d}{ds} F_c\left(e^{-a^2x^2}\right)$$

$$= \frac{1}{a\sqrt{2}} e^{-\frac{s^2}{4a^2}} \cdot \frac{s}{2a^2}$$

$$= \frac{1}{2a^3\sqrt{2}} \cdot e^{-\frac{s^2}{4a^2}}.$$

Find the F.C.T. of $e^{-a^2x^2}$ and hence find $F_s\left(x e^{-a^2x^2}\right)$.

Soln:

$$F_c(f(x)) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos sx \, dx.$$

$$F_c\left(e^{-a^2x^2}\right) = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-a^2x^2} \cos sx \, dx.$$

$$= \sqrt{\frac{2}{\pi}} \times \frac{1}{2} \int_{-\infty}^{\infty} e^{-a^2x^2} \cos sx \, dx.$$

$$= \frac{1}{\sqrt{2\pi}} \text{ R.P} \int_{-\infty}^{\infty} e^{-a^2x^2} e^{isx} \, dx.$$

$$= \frac{1}{\sqrt{2\pi}} \text{ R.P.} \int_{-\infty}^{\infty} e^{-(a^2x^2 - isx)} \, dx$$

$$a^2x^2 - isx = a^2 \left[x^2 - \frac{isx}{a^2}\right]$$

$$= a^2 \left[\underbrace{x^2 - \frac{isx}{a^2} + \left(\frac{is}{2a^2}\right)^2} - \left(\frac{is}{2a^2}\right)^2\right]$$

$$= a^2 \left[\left(x - \frac{is}{2a^2}\right)^2 - \left(\frac{is}{2a^2}\right)^2\right]$$

$$= \left[ax - \frac{is}{2a}\right]^2 + \frac{s^2}{4a^2}$$

$$= \frac{1}{\sqrt{2\pi}} R.P \int_{-\infty}^{\infty} e^{-\left[\left[ax - \frac{is}{2a}\right]^2 + \frac{s^2}{4a^2}\right]} dx.$$

$$= \frac{1}{\sqrt{2\pi}} R.P \int_{-\infty}^{\infty} e^{-\left(ax - \frac{is}{2a}\right)^2} \cdot e^{-\frac{s^2}{4a^2}} dx.$$

$$= \frac{1}{\sqrt{2\pi}} \cdot e^{-\frac{s^2}{4a^2}} R.P \int_{-\infty}^{\infty} e^{-\left(ax - \frac{is}{2a}\right)^2} dx.$$

Put $t = ax - \frac{is}{2a}.$

$dt = a\,dx.$

$$= \frac{1}{\sqrt{2\pi}} e^{-\frac{s^2}{4a^2}} R.P \int_{-\infty}^{\infty} e^{-t^2} \frac{dt}{a}.$$

$$= \frac{1}{\sqrt{2\pi}} e^{-\frac{s^2}{4a^2}} \cdot \frac{2}{a} R.P. \int_{0}^{\infty} e^{-t^2} dt.$$

Put $t^2 = u$

$2t\,dt = du.$

$dt = \frac{du}{2t}$

$= \frac{du}{2\sqrt{u}}$

$$= \frac{1}{\sqrt{2\pi}} e^{-\frac{s^2}{4a^2}} \cdot \frac{2}{a} R.P \int_{0}^{\infty} e^{-u} \frac{du}{2\sqrt{u}}.$$

$$= \frac{1}{\sqrt{2\pi}} e^{-\frac{s^2}{4a^2}} \frac{2}{2a} R.P \int_{0}^{\infty} e^{-u} u^{-\frac{1}{2}} du$$

$$= \frac{1}{\sqrt{2\pi}} \cdot \frac{2}{2a}\, e^{-\frac{s^2}{4a^2}}\, R.P \int_{0}^{\infty} e^{-u}\, u^{\frac{1}{2}-1}\, du$$

$$= \frac{1}{\sqrt{2\pi}} \cdot \frac{\cancel{2}}{\cancel{2}a}\, e^{-\frac{s^2}{4a^2}}\, R.P \sqrt{\pi}.$$

$$= \frac{1}{a\sqrt{2\pi}}\, e^{-\frac{s^2}{4a^2}}\, \cancel{\sqrt{\pi}}$$

$$= \frac{e^{-s^2/4a^2}}{a\sqrt{2}}.$$

$$F\left(e^{-a^2x^2}\right) = \frac{1}{a\sqrt{2}}\, e^{-\frac{s^2}{4a^2}}$$

$$F_s\left(x\, e^{-a^2x^2}\right) = -\frac{d}{ds} F_c\left(e^{-a^2x^2}\right)$$

$$= -\frac{d}{ds}\left(\frac{1}{a\sqrt{2}}\, e^{-\frac{s^2}{4a^2}}\right)$$

$$= -\frac{1}{a\sqrt{2}}\, e^{-\frac{s^2}{4a^2}} \left(-\frac{2s}{4a^2}\right)$$

$$= \frac{s}{[illegible]}\, e^{-\frac{s^2}{4a^2}}$$

$$= \frac{s}{a^3 2\sqrt{2}}\, e^{-\frac{s^2}{4a^2}}$$