TOPIC : 7 – HALF RANGE COSINE SERIES

Half range cosine series in the interval $(0,\pi), (0,l)$

Formula: $(0,\pi)$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

$$a_0 = \frac{2}{b-a}\int_a^b f(x)\,dx$$

$$a_n = \frac{2}{b-a}\int_a^b f(x)\cos nx\,dx$$

Formula: $(0,l)$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l}\,dx$$

$$a_0 = \frac{2}{l}\int_0^l f(x)\,dx$$

$$a_n = \frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}\,dx$$

(1) Find the half range cosine series of $f(x) = x$ in $(0,\pi)$.

Sol:

$$f(x) = x.$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} x\,dx$$

$$= \frac{2}{\pi}\left(\frac{x^2}{2}\right)_0^{\pi}$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} x\cos nx \, dx$$

$$= \frac{2}{\pi}\left[x\frac{\sin nx}{n} + \frac{\cos nx}{n^2}\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\frac{(-1)^n}{n^2} - \frac{1}{n^2}\right]$$

$$a_n = \frac{2}{n^2\pi}\left[-1+(-1)^n\right]$$

$$a_n = \begin{cases} \frac{-4}{n^2\pi} & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases}$$

$$f(x) = \frac{\pi}{2} + \sum_{n=1}^{\infty} \frac{2}{n^2\pi}\left[1-(-1)^n\right]\cos nx$$

$$f(x) = \frac{\pi}{2} + \sum_{n=odd}^{\infty} \frac{-4}{n^2\pi}\cos nx$$

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2. Find the half range cosine series of the function $f(x) = x(\pi - x)$ in $(0, \pi)$.

Sol:

$$f(x) = x\pi - x^2$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

$$a_0 = \frac{2}{\pi}\int_0^{\pi}(x\pi - x^2)\, dx$$

$$= \frac{2}{\pi}\left[\frac{x^2}{2}\pi - \frac{x^3}{3}\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\frac{\pi^3}{2} - \frac{\pi^3}{3}\right]$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} x \cos nx \, dx$$

$$= \frac{2}{\pi}\left[x\frac{\sin nx}{n} + \frac{\cos nx}{n^2} \right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\frac{(-1)^n}{n^2} - \frac{1}{n^2} \right]$$

$$a_n = \frac{2}{n^2\pi}\left[-1 + (-1)^n \right]$$

$$a_n = \begin{cases} \frac{-4}{n^2\pi} & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases}$$

$$f(x) = \frac{\pi}{2} + \sum_{n=1}^{\infty} \frac{2}{n^2\pi}\left[1-(-1)^n \right] \cos nx$$

$$f(x) = \frac{\pi}{2} + \sum_{n=odd}^{\infty} \frac{-4}{n^2\pi} \cos nx$$

2. Find the half range cosine series of the function $f(x) = x(\pi - x)$ in $(0, \pi)$.

Sol:

$$f(x) = x\pi - x^2$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} (x\pi - x^2)\, dx$$

$$= \frac{2}{\pi}\left[\frac{x^2}{2}\pi - \frac{x^3}{3} \right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\frac{\pi^3}{2} - \frac{\pi^3}{3} \right]$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} (x\pi - x^2)\cos nx \, dx$$

$u = x\pi - x^2$ $\quad \int dv = \int \cos nx \, dx$

$u_1 = \pi - 2x$ $\quad v = \frac{\sin nx}{n}$

$u_2 = -2$ $\quad v_1 = -\frac{\cos nx}{n^2}$

$u_3 = 0$ $\quad v_2 = -\frac{\sin nx}{n^3}$

$$a_n = \frac{2}{\pi}\left[(x\pi - x^2)\frac{\sin nx}{n} + (\pi - 2x)\frac{\cos nx}{n^2} + 2\frac{\sin nx}{n^3}\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[-\pi\frac{(-1)^n}{n^2} - \frac{\pi}{n^2}\right]$$

$$= \frac{2\pi}{\pi n^2}\left[-(-1)^n - 1\right]$$

$$= \frac{-2}{n^2}\left[1 + (-1)^n\right]$$

$$a_n = \begin{cases} 0 & \text{if } n \text{ is odd} \\ \frac{-4}{n^2} & \text{if } n \text{ is even} \end{cases}$$

$$f(x) = \frac{\pi^2}{6} + \sum_{n=\text{even}}^{\infty} \frac{-4}{n^2}\cos nx$$

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(3) Find the half range cosine series of $f(x) = (x-1)^2, \ 0 < x < 1$.

$$a_0 = \frac{2}{l}\int_0^1 f(x)\,dx$$

$$= 2\int_0^1 (x-1)^2\,dx$$

$$= 2\left[\frac{(x-1)^3}{3}\right]_0^1$$

$$= \frac{2}{3}\left[0+\frac{1}{1}\right]$$

$$a_0 = \frac{2}{3}.$$

$$a_n = \frac{2}{1}\int_0^1 f(x)\cos\frac{n\pi x}{l}\,dx$$

$$= 2\int_0^1 (x-1)^2 \cos\frac{n\pi x}{l}\,dx$$

$u = (x-1)^2$ $\qquad \int dv = \int \cos\frac{n\pi x}{l}\,dx$

$u_1 = 2(x-1)$ $\qquad v = \frac{\sin\frac{n\pi x}{l}}{n\pi}$

$u_2 = 2$ $\qquad v_1 = -\frac{\cos n\pi x}{n^2\pi^2}$

$\qquad v_2 = -\frac{\sin n\pi x}{n^3\pi^3}$

$$a_n = 2\left[(x-1)^2\frac{\sin n\pi x}{n\pi} + 2(x-1)\frac{\cos n\pi x}{n^2\pi^2} - 2\frac{\sin n\pi x}{n^3\pi^3}\right.$$

$$= 2\left[+\frac{2}{n^2\pi^2}\right]$$

$$= \frac{4}{n^2\pi^2}$$