TOPIC : 3 – PROBLEMS BASED ON FULL RANGE SERIES (0, 2L)

Formula for fourier series in $(0, 2l)$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

where $a_0 = \frac{2}{b-a} \int_a^b f(x)\,dx$

$$a_n = \frac{2}{b-a} \int_a^b f(x) \cos\frac{n\pi x}{l}\,dx$$

$$b_n = \frac{2}{b-a} \int_a^b f(x) \sin\frac{n\pi x}{l}\,dx.$$

Problems based on $(0, 2l)$.

1. Expand $f(x) = \begin{cases} l-x, & 0 < x \le l \\ 0, & l \le x \le 2l \end{cases}$

hence deduce that $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots = \frac{\pi}{4}$.

Sol:

$$a_0 = \frac{2}{2l} \int_0^l (l-x)\,dx$$

$$= \frac{1}{l} \left(lx - \frac{x^2}{2}\right)_0^l$$

$$= \frac{1}{l}\left(l^2 - \frac{l^2}{2}\right)$$

$$= \frac{l^2}{2l}$$

$$= l.$$

$$a_n = \frac{2}{b-a}\int_a^b f(x)\cos\frac{n\pi x}{l}\,dx$$

$$= \frac{2}{2l}\int_0^{l} (l-x)\cos\frac{n\pi x}{l}\,dx$$

$$= \frac{1}{l}\int_0^{l} (l-x)\cos\frac{n\pi x}{l}\,dx.$$

$u = l-x \qquad \int dv = \int \cos\frac{n\pi x}{l}\,dx$

$u_1 = -1 \qquad v = \dfrac{\sin\frac{n\pi x}{l}}{\frac{n\pi}{l}}$

$u_2 = 0$

$v_1 = \dfrac{-\cos\frac{n\pi x}{l}}{\left(\frac{n\pi}{l}\right)^2}$

$$a_n = \frac{1}{l}\left[(l-x)\frac{\sin\frac{n\pi x}{l}}{\frac{n\pi}{l}} - \frac{\cos\frac{n\pi x}{l}}{\left(\frac{n\pi}{l}\right)^2}\right]_0^l$$

$$= \frac{1}{l}\left[-\frac{\cos\frac{n\pi l}{l}}{\left(\frac{n\pi}{l}\right)^2} + \frac{\cos 0}{\left(\frac{n\pi}{l}\right)^2}\right]$$

$$= \frac{1}{l}\left[-\frac{(-1)^n\, l^2}{n^2\pi^2} + \frac{l^2}{n^2\pi^2}\right]$$

$$= \frac{l^2}{l\, n^2\pi^2}\left[1-(-1)^n\right]$$

$$a_n = \begin{cases} \dfrac{2l}{n^2\pi^2} & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases}$$

$$b_n = \frac{2}{2l}\int_0^l (l-x)\sin\frac{n\pi x}{l}\,dx$$

$$u = l-x \qquad \int dv = \int \sin\frac{n\pi x}{l}\,dx$$

$$u_1 = -1 \qquad v = -\cos\frac{n\pi x}{l}\Big/ n\pi/l$$

$$v_1 = -\frac{\sin\frac{n\pi x}{l}}{\left(\frac{n\pi}{l}\right)^2}$$

$$b_n = \frac{1}{l}\left[-(l-x)\frac{\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}} - \frac{\sin\frac{n\pi x}{l}}{\left(\frac{n\pi}{l}\right)^2}\right]_0^l$$

$$= \frac{1}{l}\left[l\,\frac{\cos 0}{\frac{n\pi}{l}}\right] = \frac{1}{l}\left[l\cdot\frac{l}{n\pi}\right]$$

$$b_n = \frac{l}{n\pi}.$$

$$f(x) = \frac{l}{4} + \sum_{n=odd}^{\infty} \frac{2l}{n^2\pi^2}\cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty}\frac{l}{n\pi}\sin\frac{n\pi x}{l}.$$

Deduction:

Put $x = l$ (continuous)

$$0 = \frac{l}{4} + \sum_{n=odd}^{\infty}\frac{2l}{n^2\pi^2}\cos n\pi$$

$$-\frac{l}{4} = \frac{2l}{\pi^2}\sum_{n=odd}^{\infty}\frac{(-1)^n}{n^2}$$

$$\frac{1}{4}\cdot\frac{\pi^2}{2} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots$$

$$\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\cdots=\frac{\pi^2}{8}$$

Put $x=\frac{l}{2}$ (continuous)

$$l-\frac{l}{2}=\frac{l}{4}+\sum_{n=odd}^{\infty}\frac{2l}{n^2\pi^2}\cos\frac{n\pi l}{2l}+\sum_{n=1}^{\infty}\frac{l}{n\pi}\sin\frac{n\pi . l}{2l}$$

$$\frac{l}{2}-\frac{l}{4}=\frac{2l}{\pi^2}\sum_{n=odd}^{\infty}\cos\frac{n\pi}{2}+\sum_{n=1}^{\infty}\frac{l}{n\pi}\sin\frac{n\pi}{2}$$

$$\frac{l}{4}=\sum_{n=1}^{\infty}\frac{l}{n\pi}\sin\frac{n\pi}{2}$$

$$\frac{l}{4}\cdot\frac{\pi}{l}=\frac{\sin\pi/2}{1}+\frac{\sin 2\pi/2}{2}+\frac{\sin\frac{3\pi}{2}}{3}+\frac{\sin\frac{4\pi}{2}}{4}$$

$$\frac{\pi}{4}=1+0+\frac{\sin(\pi+\frac{\pi}{2})}{3}+0+\frac{\sin(2\pi+\frac{\pi}{2})}{5}+\cdots$$

$$\frac{\pi}{4}=1-\frac{1}{3}+\frac{\sin\frac{\pi}{2}}{5}+\cdots$$

$$1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\cdots=\frac{\pi}{4}$$

(2) $f(x)=\begin{cases} x, & 0\le x\le 3 \\ 6-x, & 3\le x\le 6 \end{cases}$

Sol: $2l=6 \Rightarrow l=\frac{6}{2}=3$

$$a_0=\frac{2}{6-0}\int_0^6 f(x)\,dx.$$

$$a_0 = \frac{1}{3}\left[\int_0^3 x\,dx + \int_3^6 (6-x)\,dx\right]$$

$$= \frac{1}{3}\left[\left(\frac{x^2}{2}\right)_0^3 + \left(6x - \frac{x^2}{2}\right)_3^6\right]$$

$$= \frac{1}{3}\left[\frac{9}{2} + 36 - \cancel{\frac{36}{2}}^{18} - 18 + \frac{9}{2}\right]$$

$$= \frac{1}{3}\left[9\right]$$

$$a_0 = 3.$$

$$a_n = \frac{2}{6}\left[\int_0^3 x\cos\frac{n\pi x}{3}\,dx + \int_3^6 (6-x)\cos\frac{n\pi x}{3}\,dx\right]$$

$$= \frac{1}{3} \qquad \int dv = \int \cos\frac{n\pi x}{3}\,dx$$

$u = x$ $\qquad v = \dfrac{\sin\frac{n\pi x}{3}}{\frac{n\pi}{3}}$ $\qquad u = 6-x$

$u_1 = 1$ $\qquad\qquad\qquad u_1 = -1$

$\qquad v_1 = -\dfrac{\cos\frac{n\pi x}{3}}{\left(\frac{n\pi}{3}\right)^2}$ $\qquad u_2 = 0$

$u_2 = 0$

$$a_n = \frac{1}{3}\left[\left(x\,\frac{\sin\frac{n\pi x}{3}}{\frac{n\pi}{3}} + \frac{\cos\frac{n\pi x}{3}}{\left(\frac{n\pi}{3}\right)^2}\right)_0^3 + \left((6-x)\,\frac{\sin\frac{n\pi x}{3}}{\frac{n\pi}{3}} - \frac{\cos\frac{n\pi x}{3}}{\left(\frac{n\pi}{3}\right)^2}\right)_3^6\right]$$

$$= \frac{1}{3}\left[\frac{(-1)^n 9}{n^2\pi^2} - \frac{9}{n^2\pi^2} - \frac{9}{n^2\pi^2} + \frac{(-1)^n 9}{n^2\pi^2}\right]$$

$$a_n = \frac{2\cdot 9\cancel{3}}{\cancel{7}\cdot n^2\pi^2}\left[(-1)^n - 1\right]$$

$$= \frac{6}{n^2\pi^2}\left[(-1)^n - 1\right]$$

$$a_n = \begin{cases} \frac{-12}{n^2\pi^2}, & \text{if } n \text{ is odd} \\ 0, & \text{if } n \text{ is even} \end{cases}$$

$$b_n = \frac{2}{b-a}\int_a^b f(x)\,\sin\frac{n\pi x}{l}\,dx$$

$$= \frac{2}{6}\left[\int_0^3 x\,\sin\frac{n\pi x}{3}\,dx + \int_3^6 (6-x)\,\sin\frac{n\pi x}{3}\,dx\right]$$

$u = x \qquad \int dv = \int \sin\frac{n\pi x}{3}\,dx$

$u_1 = 1 \qquad v = \frac{-\cos\frac{n\pi x}{3}}{\frac{n\pi}{3}} \qquad u = 6-x$

$u_1 = -1$

$u_2 = 0 \qquad v_1 = \frac{-\sin\frac{n\pi x}{3}}{\left(\frac{n\pi}{3}\right)^2} \qquad u_2 = 0$

$$b_n = \frac{1}{3}\left\{\left[-x\frac{\cos\frac{n\pi x}{3}}{\frac{n\pi}{3}} + \frac{\sin\frac{n\pi x}{3}}{\left(\frac{n\pi}{3}\right)^2}\right]_0^3 + \left[-(6-x)\frac{\cos\frac{n\pi x}{3}}{\frac{n\pi}{3}} - \frac{\sin\frac{n\pi x}{3}}{\left(\frac{n\pi}{3}\right)^2}\right]_3^6\right\}$$

$$= \frac{1}{3}\left[\frac{-3\,3(-1)^n}{n\pi} + 3\,\frac{(-1)^n\cdot 3}{n\pi}\right]$$

$= \frac{1}{3}\left[0\right]$

$b_n = 0.$

$\therefore f(x) = \frac{3}{2} + \sum_{n=odd}^{\infty} \frac{-12}{n^2\pi^2} \cos\frac{n\pi x}{3}$