TOPIC7-Recurrence relations

Recurrence Relation

An equation that expresses a_n, the genera[l]
[te]rm of the sequence $\{a_n\}$ in terms of one or more
[o]f the previous terms of the sequence, namely
$a_0, a_1, \cdots, a_{n-1}$, for all integer n with $n \geq n_0$.
where n_0 is a non-negative integer is called a
recurrence relation for $\{a_n\}$ or a difference equation.

① Find the recurrence relation for the sequence
$a_n = 2n+9$, $n \geq 1$.

For $n \geq 1$, $a_n = 2n+9$

$$a_{n-1} = 2(n-1)+9 = 2n-2+9$$
$$= a_n - 2$$

$$\therefore \quad a_{n-1} = a_n - 2$$

$$\Rightarrow \quad a_n = a_{n-1} + 2, \quad \text{with } a_0 = 9$$

(2) Find the recurrence relation of

$S(n) = a^n \; ; \; n \geq 1$.

Given for $n \geq 1$, $S(n) = a^n$

Now, $S(n-1) = a^{n-1}$

$$S(n-1) = \frac{a^n}{a}$$

$$\Rightarrow \quad S(n-1) = \frac{S(n)}{a}$$

$$\Rightarrow \quad S(n) = a\,S(n-1), \text{ for } n \geq 1$$

with $S(0) = 1$

(3) Find the recurrence relation satisfying

$y_n = A\,3^n + B\,(-2)^n$.

Given $y_n = A\,3^n + B(-2)^n \longrightarrow (1)$

$y_{n+1} = A\,3^{n+1} + B\,(-2)^{n+1}$

$y_{n+1} = 3A\,3^n - 2\,B\,(-2)^n \longrightarrow (2)$

and $y_{n+2} = A\,3^{n+2} + B\,(-2)^{n+2}$

$y_{n+2} = 9A\,3^n + 4B\,(-2)^n \longrightarrow (3)$

$$y_{n+2} - y_{n+1} - 6y_n = 9A3^n - 3A\,3^n - 6A3^n + 4\,B(-2)^n + 2B(-2)^n - 6B(-2)^n$$

$$y_{n+2} - y_{n+1} - 6y_n = 0$$

Solve $a_n = 6a_{n-1} - 9a_{n-2}$ with $a_0 = 1$ and $a_1 = 6$

The recurrence relation can be rewritten as

$$a_n - 6a_{n-1} + 9a_{n-2} = 0$$

The characteristic equation is

$$r^2 - 6r + 9 = 0$$

Roots are $r = 3, 3$.

$$\therefore a_n = \alpha_1 (3)^n + \alpha_2 n (3)^n \rightarrow (1)$$

Given $a_0 = 1$

Sub. $n = 0$ in (1),

$$a_0 = \alpha_1 + \cancel{n}\alpha_2 = 1$$

$$\boxed{\alpha_1 = 1}$$

Sub. $n = 1$ in (1),

Given $a_1 = 6$

$$a_1 = \alpha_1 (3)^1 + \alpha_2 (3) = 6$$

$$3 + 3\alpha_2 = 6$$

$$\boxed{\alpha_2 = 1}$$

Sub α_1, α_2 values in (1), we get

$$a_n = 1 \cdot (3)^n + 1 \cdot n \cdot (3)^n$$

$$= 3^n + n 3^n$$

$$\boxed{a_n = (1+n) 3^n}$$

(5) Solve the recurrence relation $a_n = -3a_{n-1} - 3a_{n-2} - a_{n-3}$ given that $a_0 = 5$, $a_1 = 9$ and $a_2 = 15$.

The recurrence relation can be rewritten as

$$a_n + 3a_{n-1} + 3a_{n-2} + a_{n-3} = 0$$

The characteristic equation is

$$r^3 + 3r^2 + 3r + 1 = 0$$

$$\Rightarrow (r+1)(r+1)(r+1) = 0$$

$$\therefore \quad a_n = \alpha_1 (-1)^n + \alpha_2 n \cdot (-1)^n + \alpha_3 \cdot n^2 (-1)^n \longrightarrow (1)$$

Given $a_0 = 5$,

Put $n = 0$ in (1), we get

$$a_0 = \boxed{\alpha_1 = 5}$$

$a_1 = 9$, Put $n = 1$ in (1) we get

$$\alpha_1 (-1) + \alpha_2 (-1) + \alpha_3 (-1) = 9$$

$$-5 - \alpha_2 - \alpha_3 = 9$$

$$\Rightarrow \quad \alpha_2 + \alpha_3 = -14 \longrightarrow (2)$$

Given $a_2 = 15$, put $n = 2$ in (1), we get

$$a_2 = \alpha_1 + 2\alpha_2 + 4\alpha_3 = 15$$

$$5 + 2\alpha_2 + 4\alpha_3 = 15$$

$$\Rightarrow \quad 2\alpha_2 + 4\alpha_3 = 10$$

$$\Rightarrow \quad \alpha_2 + 2\alpha_3 = 5 \longrightarrow (3)$$

solving ② and ③,

② - ③ $\Rightarrow$ $\alpha_3 - 2\alpha_3 = -14 - 5$

$$-\alpha_3 = -19$$

$$\boxed{\alpha_3 = 19}$$

sub. in ③,

$$\alpha_2 + 38 = 5 \Rightarrow \boxed{\alpha_2 = -33}$$

sub. α_1, α_2 and α_3 values in ①,

$$a_n = 5(-1)^n - 33n(-1)^n + 19n^2(-1)^n$$

$$\boxed{a_n = \left[5 - 33n + 19n^2\right](-1)^n}$$

⑥ Find the solution to the recurrence relation $a_n = 6a_{n-1} - 11a_{n-2} + 6a_{n-3}$ with the initial condi. $a_0 = 2$, $a_1 = 5$ and $a_2 = 15$.

.

.