TOPIC : 2 - TAUTOLOGY & LOGICAL EQUIVALENCE

Tautology

A statement which is true always irrespective of the truth values of the individual variables is called a tautology.

Example $P \vee \neg P$ is a Tautology.

Contradiction

A statement which is always false is called a contradiction.

Example $P \wedge \neg P$ is a contradiction.

Contigency

A statement which is neither Tautology nor contradiction is called contigency.

① Show that $Q \vee (P \wedge \neg Q) \vee (\neg P \wedge \neg Q)$ is a tautology.

P	Q	$\neg P$	$\neg Q$	$P \wedge \neg Q$	$Q \vee (P \wedge \neg Q)$	$\neg P \wedge \neg Q$	S
T	T	F	F	F	T	F	T
T	F	F	T	T	T	F	T
F	T	T	F	F	T	F	T
F	F	T	T	F	F	T	T

$\therefore$ Given statement is Tautology.

③ Show that $(P \to Q) \wedge (Q \to R) \to (P \to R)$ is a tautology.

P	Q	R	$P \to Q$	$Q \to R$	$(P \to Q) \wedge (Q \to R)$	$P \to R$	S
T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	T
T	F	T	F	T	F	T	T
T	F	F	F	T	F	F	T
F	T	T	T	T	T	T	T
F	T	F	T	F	F	T	T
F	F	T	T	T	T	T	T
F	F	F	T	T	T	T	T

Hence the given statement is a tautology.

④ Prove $((p \vee q) \wedge \neg(\neg p \wedge (\neg q \vee \neg r))) \vee (\neg p \wedge \neg q) \vee (\neg p \wedge \neg r)$ is a tautology.

p	q	r	$\neg p$ (1)	$\neg q$ (2)	$\neg r$ (3)	$p \vee q$ (4)	$\neg q \vee \neg r$ (5)	$\neg p \wedge \neg q \vee \neg r$ (6)	$\neg(\neg p \wedge \neg q \vee \neg r)$ (7)	$(4) \wedge (7)$ (8)	$(1) \wedge (2)$ (9)	$(8) \vee (9)$	$(1) \wedge (3)$ (10)	$(9) \vee (10)$
T	T	T	F	F	F	T	F	F	T	T	F	T	F	T
T	T	F	F	F	T	T	T	F	T	T	F	T	F	T
T	F	T	F	T	F	T	T	F	T	T	F	T	F	T
T	F	F	F	T	T	T	T	F	T	T	F	T	F	T
F	T	T	T	F	F	T	F	F	T	T	F	T	F	T
F	T	F	T	F	T	T	T	T	F	F	F	F	T	T
F	F	T	T	T	F	F	T	T	F	F	T	T	F	T
F	F	F	T	T	T	F	T	T	F	F	T	T	T	T

Equivalence

Two statements P and Q are equivalent iff $P \leftrightarrow Q$ or $P \rightleftarrows Q$ is a tautology. It is denoted by the symbol $P \Leftrightarrow Q$ which is read as "P is equivalent to Q".

Logical Equivalences (or) Equivalence rules	
Idempotent Laws	$p \wedge p \Leftrightarrow p$ $p \vee p \Leftrightarrow p$
Associative Laws	$(p \wedge q) \wedge r \Leftrightarrow p \wedge (q \wedge r)$ $(p \vee q) \vee r \Leftrightarrow p \vee (q \vee r)$
Commutative Laws	$p \wedge q \Leftrightarrow q \wedge p$ $p \vee q \Leftrightarrow q \vee p$
De Morgan's Laws	$\neg(p \wedge q) \Leftrightarrow (\neg p \vee \neg q)$ $\neg(p \vee q) \Leftrightarrow (\neg p \wedge \neg q)$
Distributive Laws	$p \wedge (q \vee r) \Leftrightarrow (p \wedge q) \vee (p \wedge r)$ $p \vee (q \wedge r) \Leftrightarrow (p \vee q) \wedge (p \vee r)$
Complement Laws	$p \wedge \neg p \Leftrightarrow F$ $p \vee \neg p \Leftrightarrow T$
Absorption Laws	$p \vee (p \wedge q) \Leftrightarrow p$ $p \wedge (p \vee q) \Leftrightarrow p$
contrapositive Law	$p \to q \Leftrightarrow \neg q \to \neg p$
Conditional as disjunction	$p \to q \Leftrightarrow \neg p \vee q$
Biconditional as conditional	$p \leftrightarrow q \Leftrightarrow (p \to q) \wedge (q \to p)$

E ① Show that $(\neg P \wedge (\neg Q \wedge R)) \vee (Q \wedge R) \vee (P \wedge R) \Leftrightarrow$ R

$(\neg P \wedge (\neg Q \wedge R)) \vee (Q \wedge R) \vee (P \wedge R)$

$\Leftrightarrow (\neg P \wedge (\neg Q \wedge R)) \vee ((Q \vee P) \wedge R)$ (∵ Distributive law)

$\Leftrightarrow ((\neg P \wedge \neg Q) \wedge R) \vee ((Q \vee P) \wedge R)$ (∵ Associative law)

$\Leftrightarrow [(\neg P \wedge \neg Q) \vee (Q \vee P)] \wedge R$ (∵ Distributive law)

$\Leftrightarrow [\neg (P \vee Q) \vee (P \vee Q)] \wedge R$ (∵ De morgan's law)

$\Leftrightarrow T \wedge R$ $[\because P \vee \neg P \Leftrightarrow T]$

$\Leftrightarrow R$ $[\because P \wedge T \Leftrightarrow P]$

(8) Show that $(P \vee Q) \wedge (\neg P \wedge (\neg P \wedge Q)) \Leftrightarrow (\neg P$

$(P \vee Q) \wedge (\neg P \wedge (\neg P \wedge Q))$

$\Leftrightarrow (P \vee Q) \wedge ((\neg P \wedge \neg P) \wedge Q)$ [∵ Associative law]

$\Leftrightarrow (P \vee Q) \wedge (\neg P) \wedge Q)$ [∵ Idemponent law]

$\Leftrightarrow (P \vee (\neg P \wedge Q)) \wedge (Q \vee (\neg P \wedge Q))$

[∵ Distributive law]

$\Leftrightarrow ((P \wedge \neg P) \wedge Q) \wedge (Q \vee (\neg P \wedge Q))$

[∵ Associative law]

$\Leftrightarrow (T \wedge Q) \wedge (Q \vee (Q \wedge \neg P))$ [∵ Commutative]

$\Leftrightarrow Q \wedge ((Q \vee Q) \wedge \neg P)$ [∵ Associative]

$\Leftrightarrow Q \wedge (Q \wedge \neg P)$ [∵ Idempotent]

$\Leftrightarrow (Q \wedge Q) \wedge \neg P$ [∵ Associative]

$\Leftrightarrow Q \wedge \neg P$ [∵ Idempotent]

Tautological Implication

A statement P is said to be tautologically imply a statement Q iff $P \rightarrow Q$ is a tautology. We shall denote this idea by $P \Rightarrow$

(2) Prove that $(P \rightarrow Q) \wedge (R \rightarrow Q) \Rightarrow (P \vee R) \rightarrow Q$

T.S.T $(P \rightarrow Q) \wedge (R \rightarrow Q) \rightarrow ((P \vee R) \rightarrow Q)$ is a tautology.

$(P \rightarrow Q) \wedge (R \rightarrow Q) \rightarrow ((P \vee R) \rightarrow Q)$

$\Leftrightarrow (\neg P \vee Q) \wedge (\neg R \vee Q) \rightarrow (\neg (P \vee R) \vee Q)$ [∵ $P \rightarrow Q \Leftrightarrow \neg P \vee Q$]

$\Leftrightarrow ((\neg P \wedge \neg R) \vee Q) \rightarrow (\neg (P \vee R) \vee Q)$ [∵ Distributive

$\Leftrightarrow (\neg (P \vee R) \vee Q) \rightarrow (\neg (P \vee R) \vee Q)$ [Demorgan's law

$\Leftrightarrow \neg (\neg (P \vee R) \vee Q) \vee (\neg (P \vee R) \vee Q)$ [∵ $P \rightarrow Q \Leftrightarrow \neg P \vee Q$]

$\Leftrightarrow T$ [$\neg P \vee P \Leftrightarrow T$]

(3) Show that $((P\vee Q)\wedge \neg(\neg P\wedge(\neg Q\vee \neg R)))\vee(\neg P\wedge \neg Q)\vee(\neg P\wedge \neg R)$ is a tautology.

Consider $(P\vee Q)\wedge \neg(\neg P\wedge(\neg Q\vee \neg R))$

$\Leftrightarrow (P\vee Q)\wedge \neg(\neg P\wedge \neg(Q\wedge R))$ [∵ Demorgan's Law]

$\Leftrightarrow (P\vee Q)\wedge \neg(\neg(P\vee(Q\wedge R)))$ [∵ Demorgan's law]

$\Leftrightarrow (P\vee Q)\wedge(P\vee(Q\wedge R))$ [∵ Double negation]

$\Leftrightarrow \cancel{(P\vee Q)\wedge}\,((P\vee Q)\wedge(P\vee R))$ [∵ Distributive law]

$\Leftrightarrow P\vee(Q\wedge(Q\wedge R))$

$\Leftrightarrow P\vee((Q\wedge Q)\wedge R)$ [∵ Associative law]

$\Leftrightarrow P\vee(Q\wedge R)$ [∵ Idempotent law]

Now

$(\neg P\wedge \neg Q)\vee(\neg P\wedge \neg R)$

$\Leftrightarrow \neg(P\vee Q)\vee \neg(P\vee R)$ [Demorgan's law]

$\Leftrightarrow \neg((P\vee Q)\wedge(P\vee R))$ [Demorgan's law]

$\Leftrightarrow \neg(P\vee(Q\wedge R))$ [∵ Distributive law]

Now $(P\vee(Q\wedge R))\vee \neg(P\vee(Q\wedge R))\Leftrightarrow T$

Hence the gn. equation is a tautology.