TOPIC:1.-Mathematical Induction

Mathematical Induction

Principle of Mathematical Induction

Let $P(n)$ be a statement or proposition involving the natural number 'n', then we complete two steps:

Basic step: we must prove that $P(1)$ is true.

Inductive step: By assuming $P(k)$ is true, we must prove that $P(k+1)$ is also true.

1. Use mathematical induction to show that $n! \geq 2^{n+1}$, $n = 5, 6, \ldots$

Let $P(n) : n! \geq 2^{n+1}$, $n = 5, 6, \ldots$

(i) $P(5) : 5! \geq 2^6 = 120 \geq 64$ is true

Assume $P(k) : k! \geq 2^{k+1}$ is true. $\rightarrow (1)$

(ii) claim : $P(k+1)$ is true.

By (1), $k! \geq 2^{k+1}$ Multiply both sides by 2

$2k! \geq 2^{k+2}$

$\Rightarrow \quad (k+1)\, k! \geq 2^{k+2}$

$\Rightarrow \quad (k+1)! \geq 2^{k+2}$

$\therefore$ $P(k+1)$ is true.

Hence, by the principle of mathematical induction,

$$n! \geq 2^{n+1}, \quad \text{for } n = 5, 6, \cdots$$

② Use Mathematical induction, show that

$$1+2+3+\cdots+n = \frac{n(n+1)}{2}.$$

Let $P(n)$: $1+2+3+\cdots+n = \dfrac{n(n+1)}{2}$

(i) $P(1)$: $1 = \dfrac{1(2)}{2}$ is true.

(ii) Assume $P(k)$: $1+2+3+\cdots+k = \dfrac{k(k+1)}{2}$ is true $\longrightarrow$ ①

(iii) claim : $P(k+1)$ is true.

$P(k+1)$: $1+2+3+\cdots+k+k+1$

$= \dfrac{k(k+1)}{2} + k+1$ (by ①)

$$= \frac{K(K+1) + 2K+2}{2} = \frac{K(K+1) + 2(K+1)}{2}$$

$$= \frac{(K+1)(K+2)}{2} \quad \text{which is true.}$$

By the principle of mathematical induction,

$$1+2+3+\cdots+n = \frac{n(n+1)}{2}$$

③ Prove that by the principle of mathematical induction, for 'n' a positive integer,

$$1^2+2^2+3^2+\cdots+n^2 = \frac{n(n+1)(2n+1)}{6}.$$

Let $P(n) = 1^2+2^2+\cdots+n^2 = \frac{n(n+1)(2n+1)}{6}$

(i) $P(1)$: $1^2 = \frac{1(1+1)(2+1)}{6}$ is true.

(ii) Assume $P(K)$: $1^2+2^2+\cdots+K^2 = \frac{K(K+1)(2K+1)}{6}$ is true $\rightarrow (1)$

(iii) claim : $P(K+1)$ is true.

$P(K+1)$: $1^2+2^2+\cdots+K^2+(K+1)^2$

$$= \frac{K(K+1)(2K+1)}{6} + (K+1)^2 \quad \text{by } (1)$$

$$= \frac{K(K+1)(2K+1) + 6(K+1)^2}{6}$$

$$= \frac{(K+1)\left(K(2K+1) + 6(K+1)\right)}{6}$$

$$= \frac{(K+1)(K+2)(2K+3)}{6}$$

$$= \frac{(K+1)\left((K+1)+1\right)\left(2(K+1)+1\right)}{6}$$

$\therefore$ $P(K+1)$ is true.

$\therefore$ By the principle of mathematical induction,

$$1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6} \quad \forall n$$