TOPIC 10-GENERATING FUNCTIONS:

Generating Function

The generating function for the sequence with terms $a_0, a_1, \ldots, a_n$ of real numbers is the infinite sum.

$$G(x) = G(S, x) = a_0 + a_1 x + \cdots + a_n x^n + \cdots$$

$$= \sum_{n=0}^{\infty} a_n x^n$$

.

1) Using generating function, solve the recurrence relation $a_n = 3a_{n-1}$ for $n \geq 1$ with $a_0 = 2$.

Let $G(x) = \sum_{n=0}^{\infty} a_n x^n$ be the generating function of the sequence $\{a_n\}$

Given recurrence relation can be written as

$$a_n - 3a_{n-1} = 0$$

Multiplying the above equation by x^n and summing from 1 to ∞, we get

.

$$\Rightarrow \sum_{n=1}^{\infty} a_n x^n - \sum_{n=1}^{\infty} 3 a_{n-1} x^n = 0$$

$$\Rightarrow \sum_{n=1}^{\infty} a_n x^n - 3x \sum_{n=1}^{\infty} a_{n-1} x^{n-1} = 0$$

$$\Rightarrow \left(G(x) - a_0 \right) - 3x\, G(x) = 0$$

$$\Rightarrow G(x) \left[1 - 3x \right] = 2$$

$$G(x) = \frac{2}{1-3x} = 2(1-3x)^{-1}$$

$$= 2 \left[1 + 3x + (3x)^2 + \cdots \right]$$

$$G(x) = 2 \sum_{n=0}^{\infty} 3^n x^n$$

$$\Rightarrow \sum_{n=0}^{\infty} a_n x^n = 2 \sum_{n=0}^{\infty} 3^n x^n$$

$$\Rightarrow \boxed{a_n = 2 \cdot 3^n}$$

② Solve the recurrence relation $S(n+1) - 2S(n) = 4^n$, $S(0) = 1$, $n \geq 0$ by use generating function.

The recurrence relation can be written as

$$a_{n+1} - 2a_n - 4^n = 0$$

Multiplying the above equation by x^n and summing from 0 to ∞, we have

$$\sum_{n=0}^{\infty} a_{n+1} x^n - 2\sum_{n=0}^{\infty} a_n x^n - \sum_{n=0}^{\infty} 4^n x^n = 0$$

$$\Rightarrow \frac{1}{x}\sum_{n=0}^{\infty} a_{n+1} x^{n+1} - 2\sum_{n=0}^{\infty} a_n x^n - \sum_{n=0}^{\infty} 4^n x^n = 0$$

$$\Rightarrow \frac{1}{x}\left[G(x) - a_0\right] - 2G(x) - \left[1 + 4x + (4x)^2 + \cdots\right] = 0$$

$$\Rightarrow \frac{1}{x}\left[G(x) - a_0\right] - 2G(x) - \frac{1}{1-4x} = 0$$

$$\Rightarrow \frac{1}{x}\left[G(x) - a_0\right] - 2G(x) = \frac{1}{1-4x}$$

$$\Rightarrow \frac{1}{x}\left[G(x) - 1\right] - 2G(x) = \frac{1}{1-4x}$$

$$\Rightarrow G(x)\left[\frac{1}{x}-2\right] = \frac{1}{1-4x}+\frac{1}{x}$$

$$= \frac{x+1-4x}{x(1-4x)} = \frac{1-3x}{x(1-4x)}$$

$$\Rightarrow G(x) = \frac{1-3x}{1-4x}\,\frac{1}{1-2x}$$

$$\frac{1-3x}{(1-2x)(1-4x)} = \frac{A}{1-2x}+\frac{B}{1-4x} \rightarrow (1)$$

$$1-3x = A(1-4x)+B(1-2x)$$

Put $x=\frac{1}{4}$, $1-\frac{3}{4} = B\left(1-\frac{2}{4}\right)$

$$\frac{1}{4} = \frac{1}{2}B \Rightarrow \boxed{B=\frac{1}{2}}$$

Put $x = \frac{1}{2}$, $1 - \frac{3}{2} = A(1-2)$

$$\boxed{A = \frac{1}{2}}$$

$$G(x) = \frac{\frac{1}{2}}{1-2x} + \frac{\frac{1}{2}}{1-4x}$$

$$= \frac{1}{2}(1-2x)^{-1} + \frac{1}{2}(1-4x)^{-1}$$

$$= \frac{1}{2}\left[1 + 2x + (2x)^2 + \cdots\right] + \frac{1}{2}\left[1 + 4x + (4x)\right.$$

$$= \frac{1}{2}\sum_{n=0}^{\infty} 2^n x^n + \frac{1}{2}\sum_{n=0}^{\infty} 4^n x^n$$

$$a_n = \frac{2^n}{2} + \frac{4^n}{2}$$

$$a_n = 2^{n-1} + 2(4)^{n-1}$$