The Principle of Inclusion - Exclusion

Assume two tasks T_1 and T_2 that can be done at the same time. Now, to find the number of ways to do one of the two tasks T_1 and T_2, if we add number ways to do each task then it leads to an over count, since the ways to do both tasks are counted twice. To correctly count the number of ways to do one of the two tasks, we add the number of ways to do both tasks.

ii) $n(T_1 \cup T_2) = n(T_1) + n(T_2) - n(T_1 \cap T_2)$

This technique is called the principle of Inclusion - Exclusion.

(1) Find the number of integers between 1 and 250 both inclusive that are divisible by any of the integers 2, 3, 5, 7.

Let A, B, C and D denote the number of positive integers between 1 - 250 that are divisible by 2, 3, 5 and 7 resply.

Now, $|A| = \left[\frac{250}{2}\right] = 125$

$|B| = \left[\frac{250}{3}\right] = 83$

$|C| = \left[\frac{250}{5}\right] = 50$

$|D| = \left[\frac{250}{7}\right] = 35$

$|A \cap B| = \left[\frac{250}{2 \times 3}\right] = 41$

$|A \cap C| = \left[\frac{250}{2 \times 5}\right] = 25$

$|A \cap C \cap D| = \left[\frac{250}{2 \times 5 \times 7}\right] = 3$

$|B \cap C \cap D| = \left[\frac{250}{3 \times 5 \times 7}\right] = 2$

$|A \cap B \cap C \cap D| = \left[\frac{250}{2 \times 3 \times 5 \times 7}\right] = 1$

By the principle of Inclusion & Exclusion,

$|A \cap D| = \left[\frac{250}{2 \times 7}\right] = 17$

$|B \cap C| = \left[\frac{250}{3 \times 5}\right] = 16$

$|B \cap D| = \left[\frac{250}{3 \times 7}\right] = 11$

$|C \cap D| = \left[\frac{250}{5 \times 7}\right] = 7$

$|A \cap B \cap C| = \left[\frac{250}{2 \times 3 \times 5}\right] = 8$

$|A \cap B \cap D| = \left[\frac{250}{2 \times 3 \times 7}\right] = 5$

$$|A \cup B \cup C \cup D|$$

$$= |A| + |B| + |C| + |D| - |A \cap B| - |A \cap C| - |A \cap D|$$
$$- |B \cap D| - |B \cap C| - |C \cap D| + |A \cap B \cap C|$$
$$+ |A \cap B \cap D| + |A \cap C \cap D| + |B \cap C \cap D|$$
$$- |A \cap B \cap C \cap D|$$

$$= (125 + 83 + 50 + 35) - (41 + 25 + 17 + 16 + 11 + 7)$$
$$+ (8 + 5 + 3 + 2) - 1$$

$$= 193$$

(2) Find the number of integers between 1 and 500 that are not divisible by any of the integers 2, 3, 5 and 7.

Let A, B, C, D denote the number of positive integers between 1 and 500 that are divisible by 2, 3, 5 and 7 resply.

$$|A| = \left[\frac{500}{2}\right] = 250$$

$$|B| = \left[\frac{500}{3}\right] = 166$$

$$|C| = \left[\frac{500}{5}\right] = 100$$

$$|D| = \left[\frac{500}{7}\right] = 71$$

By the P.I.E.,

$|A \cup B \cup C \cup D|$

$= (250 + 166 + 100 + 71) - (83 + 50 + 35 + 33 + 23 + 14)$

$+ (16 + 11 + 7 + 4) - 2$

$= 385$

∴ The number of positive integers between 1 and 500 not divisible by 2, 3, 5 and 7.

$=$ Total $- 385 = 500 - 385$

$= 115$.