Problem 1: Identify Irreducibility

Determine if the polynomial $p(x) = x^3 - 2$ is irreducible over $\mathbb{Q}$.

Hint: Use the Rational Root Theorem to check for rational roots.

Problem 2: Factorization in Finite Fields

Factor the polynomial $q(x) = x^4 + x^3 + x^2 + x + 1$ over the finite field $\mathbb{F}_2$.

Hint: Check for linear factors and use polynomial long division if necessary.

Problem 3: Degree Condition

Show that any polynomial of degree 1 is irreducible over any field.

Solution: A polynomial of the form $ax + b$ (where $a \neq 0$) cannot be factored further, as the only factors would be constant polynomials.

Problem 4: Finding Irreducible Polynomials

Find an irreducible polynomial of degree 3 over $\mathbb{F}_5$.

Hint: List polynomials of degree 3 and check for roots in $\mathbb{F}_5$ (0, 1, 2, 3, 4).

Problem 5: Showing a Polynomial is Reducible

Prove that the polynomial $r(x) = x^4 + 1$ is reducible over $\mathbb{R}$.

Hint: Show that $r(x)$ can be factored into two quadratics or check for real roots.

Problem 6: Constructing Irreducible Polynomials

Construct an irreducible polynomial of degree 5 over $\mathbb{Z}$ or $\mathbb{Q}$.