TOPIC : 4 - **Parseval's identity**

Example 4:-

Show that the transform of $e^{-\frac{x^2}{2}}$ is $e^{-\frac{s^2}{2}}$ by finding the Fourier transform of $e^{-a^2x^2}$, $a>0$.

Soln:-

self reciprocal:-

If the Fourier transform of $f(x)$ is $f(s)$, then $f(x)$ is said to be self reciprocal under Fourier transform

F.T. of $f(x) = e^{-a^2x^2}$ is.

$$F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-a^2x^2} e^{isx}\, dx,$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-(a^2x^2 - isx)} dx.$$

$$a^2x^2 - isx = \left[(ax)^2 - 2ax\left(\frac{is}{2a}\right) + \left(\frac{is}{2a}\right)^2\right] + \frac{s^2}{4a^2}$$

$$= \left[ax - \frac{is}{2a}\right]^2 + \frac{s^2}{4a^2}$$

$$\therefore F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{s^2}{4a^2}} \cdot e^{-\left(ax - \frac{is}{2a}\right)^2} dx.$$

$$= \frac{e^{-\frac{s^2}{4a^2}}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\left[ax - \frac{is}{2a}\right]^2} dx.$$

Put $u = ax - \frac{is}{2a}$

$du = a\,dx.$

$$\therefore F(s) = \frac{1}{a\sqrt{2\pi}} e^{-\frac{s^2}{4a^2}} \int_{-\infty}^{\infty} e^{-u^2} du = \frac{2e^{-s^2/4a^2}}{a\sqrt{2\pi}} \int_0^{\infty} e^{-u^2} du.$$

Put $t = u^2$, $du = \frac{dt}{2\sqrt{t}}$.

$$\therefore F(s) = \frac{2e^{-s^2/4a^2}}{a\sqrt{2\pi}} \int e^{-t} \frac{dt}{2\sqrt{t}} = \frac{e^{-\frac{s^2}{4a^2}}}{a\sqrt{2\pi}} \int_0^{\infty} e^{-t} t^{\frac{1}{2}-1} dt$$

$$F(e^{-a^2x^2}) = \frac{e^{-\frac{s^2}{4a^2}}}{a\sqrt{2\pi}} \Gamma_{\frac{1}{2}} = \frac{e^{-\frac{s^2}{4a^2}}}{a\sqrt{2}} \quad \left[\because \Gamma_{\frac{1}{2}} = \sqrt{\pi}\right]$$

$\longrightarrow (1).$

Putting $a = \frac{1}{\sqrt{2}}$ in ①, we have.

$$F\left(e^{-x^2/2}\right) = \frac{1}{\frac{1}{\sqrt{2}}\cdot\sqrt{2}} \cdot e^{-\frac{s^2}{4\cdot\frac{1}{2}}} = e^{-\frac{s^2}{2}}.$$

$$\therefore F\left\{e^{-\frac{x^2}{2}}\right\} = e^{-\frac{s^2}{2}}$$

The function $f(x) = e^{-x^2/2}$ is self-reciprocal under Fourier transform