TOPIC : 8 – Properties of Fourier Transform

Properties of Fourier Transform:-

1. Linear Property:

$$F\{af(x)+bg(x)\} = aF(s)+bG(s).$$

Proof:

$$F(af(x)+bg(x)) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{isx}[af(x)+bg(x)]dx$$

$$= a\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{isx}f(x)dx\right]+b\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{isx}g(x)dx\right]$$

$= aF(s)+bG(s)$, where $F(s)$ and $G(s)$ are the Fourier transforms of $f(x)$ and $g(x)$.

2. Shifting Property:

If $F(f(x)) = F(s)$, then $F[f(x-a)] = e^{isa}F(s)$

Proof:

$$F(f(x-a)) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{isx}f(x-a)dx.$$

Put $t = x - a \Rightarrow dt = dx$.

$$\therefore F[f(x-a)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{is[t+a]} f(t)\,dt.$$

$$= e^{isa} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ist} f(t)\,dt$$

$$= e^{isa} \cdot F(s).$$

3. change of scale property.

If $F(f(x)) = F(s)$, then $F\{f(ax)\} = \frac{1}{|a|} F\left(\frac{s}{a}\right)$ where $a \neq 0$.

Soln:

$$F\{f(ax)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{isx} f(ax)\,dx.$$

Put $t = ax$

$dt = a\,dx \Rightarrow dx = \frac{dt}{a}$.

$$\therefore F\{f(ax)\} = \frac{1}{a}\left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{it\left(\frac{s}{a}\right)} f(t)\,dt\right].$$

By definition,

$$F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ist} f(t)\,dt$$

$$\therefore F\{f(ax)\} = \frac{1}{a} F\left(\frac{s}{a}\right), a>0$$

$$\text{and} = -\frac{1}{a} F\left(\frac{s}{a}\right), a<0.$$

$$\therefore F\{f(ax)\} = \frac{1}{|a|} F\left(\frac{s}{a}\right) \text{ where } a \neq 0.$$

4. If $F\{f(x)\} = F(S)$, then $F\{e^{iax} f(x)\} = F(S+a)$

Proof:

$$F\{e^{iax} f(x)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{iax} \cdot e^{isx} f(x)\, dx.$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i(s+a)x} f(x)\, dx.$$

$$\therefore F\{e^{iax} f(x)\} = F(s+a).$$

Property 5: Modulation theorem.

If $F\{f(x)\} = F(s)$, then $F\{f(x).\cos ax\} = \frac{1}{2}\left[F(s-a) + F(s+a)\right]$.

Proof:

$$F\{f(x)\} = F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx.$$

$$F\{f(x).\cos ax\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{isx} \cos ax\, f(x)\, dx.$$

But $\cos ax = \dfrac{e^{iax} + e^{-iax}}{2}$.

$$\therefore F\{f(x)\cos ax\} = \frac{1}{2}\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{isx}\left(e^{iax}+e^{-iax}\right) f(x)\,dx\right]$$

$$= \frac{1}{2}\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{i(s+a)x} f(x)\,dx\right] + \frac{1}{2}\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{i(s-a)x} f(x)\,dx\right]$$

$$= \frac{1}{2}\left[F(s+a)+F(s-a)\right].$$

Theorem 6:

If $F\{f(x)\} = F(s)$, then $F\{x^n f(x)\} = (-i)\dfrac{d^n}{ds^n}\{F(s)\}$.

Property 7:

If $F\{f(x)\} = F(s)$, then $F\{f^n(x)\} = (-is)^n F(s)$ if $f(x), f'(x), \ldots f^{n-1}(x)$ all tend to zero as $x \to \pm\infty$.

where $f^n(x) = \dfrac{d^n}{dx^n}(f(x))$.

Proof:

$$F\{f'(x)\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f'(x)\, e^{isx}\,dx.$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{isx}\, d(f(x))$$

$$= \frac{1}{\sqrt{2\pi}}\left\{\left(f(x)\,e^{isx}\right)_{-\infty}^{\infty} - \int_{-\infty}^{\infty} f(x)\,e^{isx}(is)\,dx\right\}$$

Since $f(x) \to 0$ as $x \to \infty$, we have.

$$F\{f'(x)\} = (-is)\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx\right] = (-is)F(s) \quad \text{(1)}$$

lly

$$F\{f''(x)\} = \frac{1}{\sqrt{2\pi}}\left[f'(x)\, e^{isx}\right]_{-\infty}^{\infty} - \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{isx}(is)f'(x)\, dx$$

$$= (-is)\, F\{f'(x)\}$$

$$F\{f''(x)\} = (-is)^2 F(s) \quad \text{using (1)}.$$

In general,

$$F\{f^n(x)\} = (-is)^n F(s) \text{ provided}$$

$f, f', \ldots, f^{n-1}$ all tends to 0 as $x \to \pm\infty$.

Property 8 : (M)

If $F\{f(x)\} = F(s)$ then $F\left\{\int_a^x f(x)\,dx\right\} = \frac{F(s)}{(-is)}$

Proof: Let $\phi(s) = \int_a^x f(x)\,dx$.

$$\phi'(s) = f(x).$$

Also $F(\phi'(x)) = (-is)\,F\{\phi(x)\}$

$$= (-is)\,F\left\{\int_a^x f(x)\,dx\right\}$$

$$F\left[\int_a^x f(x)\,dx\right] = \frac{F(\phi'(x))}{-is} = \frac{F\{f(x)\}}{-is}$$

$$\therefore F\left[\int_a^x f(x)\,dx\right] = \frac{F(s)}{-is}.$$