TOPIC: 7 - Fourier sine Transforms

Fourier sine and Cosine Transform.

Fourier sine transform.

The infinite F.S.T. of $f(x)$ is defined by

$$F_s\{f(x)\} = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin sx \, dx.$$

It is also defined by $\overline{f_s}(s)$ or $F_s(s)$.

The inverse Fourier sine transform of $\overline{f}_s(s)$ is defined by

$$f(x) = F_s^{-1}\{\overline{f}_s(s)\} = \sqrt{\frac{2}{\pi}} \int_0^\infty \overline{f}_s(s) \sin sx \, ds.$$

Fourier Cosine Transform.

The infinite F.C.T. of $f(x)$ is defined by

$$F_c\{f(x)\} = \overline{f}_c(s) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos sx \, dx$$

The inverse F.C.T. of $\overline{f}_c(s)$ is defined by

$$f(x) = F_c^{-1}\{\overline{f}_c(s)\} = \sqrt{\frac{2}{\pi}} \int_0^\infty \overline{f}_c(s) \cos sx \, ds$$

Properties of F.C.T & F.S.T.

1. Cosine and sine transforms are linear

If $F_c\{f(x)\} = \bar{f}_c(s)$ and $F_c\{g(x)\} = \bar{g}_c(s)$

then $F_c\{af(x)+bg(x)\} = a\bar{f}_c(s) + b\bar{g}_c(s)$

$= aF_c\{f(x)\} + bF_c\{g(x)\}$

Identities:-

If $F_c(s)$ and $G_c(s)$ are the F.C.Ts and $F_s(s)$, $G_s(s)$ are the F.S.Ts of $f(x)$ and $g(x)$ respectively, then

1. $\int_0^\infty f(x)\cdot g(x)\,dx = \int_0^\infty F_c(s)\cdot G_c(s)\,ds.$

2. $\int_0^\infty f(x)\cdot g(x)\,dx = \int_0^\infty F_s(s)\, G_s(s)\,ds.$

3. $\int_0^\infty |f(x)|^2\,dx = \int_0^\infty |F_c(s)|^2\,ds = \int_0^\infty |F_s(s)|^2\,ds.$

Example 8: Find the F.C. and F.S. transforms of e^{-ax}, $a>0$ and hence deduce the inversion formula.

Soln:

$$F_s(e^{-ax}) = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-ax} \sin sx \, dx.$$

$$= \sqrt{\frac{2}{\pi}} \left\{ \frac{e^{-ax}}{a^2+s^2}(-a \sin sx - s\cos sx) \right\}_0^\infty$$

$$= \sqrt{\frac{2}{\pi}} \frac{s}{a^2+s^2}, \text{ if } a>0 \rightarrow (*).$$

$$F_c(e^{-ax}) = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-ax} \cos sx \, dx.$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{e^{-ax}}{a^2+s^2} (-a \cos sx + s \sin sx) \right]_0^\infty$$

By inversion formula of (*),

$$e^{-ax} = \sqrt{\frac{2}{\pi}} \int_0^\infty \left(\sqrt{\frac{2}{\pi}} \frac{s}{a^2+s^2} \cdot \sin sx \right) ds.$$

$$\therefore \int_0^\infty \frac{s}{a^2+s^2} \sin sx \, dx = \frac{\pi}{2} e^{-ax}, \quad a>0.$$

($s \to x$, $x \to \alpha$) changing the variables.

$$\int_0^\infty \frac{x \sin \alpha x}{a^2+x^2}\, dx = \frac{\pi}{2} e^{-\alpha a},\ a>0 \quad \text{--- (3)}$$

By inversion formula of (2),

$$e^{-ax} = \sqrt{\frac{2}{\pi}} \int_0^\infty \left(\sqrt{\frac{2}{\pi}} \cdot \frac{a}{a^2+s^2} \right) \cos sx\, ds$$

$$\int_0^\infty \frac{\cos sx}{a^2+s^2}\, ds = \frac{\pi}{2a} e^{-ax}.$$

Changing the variables,

$$\int_0^\infty \frac{\cos \alpha x}{a^2+x^2}\, dx = \frac{\pi}{2a} \cdot e^{-\alpha a},\ a>0. \quad \text{--- (4)}$$

Example 9:-

Find F.S.T. of $\frac{x}{a^2+x^2}$ and F.C.T. of $\frac{1}{a^2+x^2}$.

Soln:

$$F_s\left(\frac{x}{a^2+x^2}\right) = \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{x}{a^2+x^2} \sin sx\, dx.$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{\pi}{2} e^{-as} \right] \quad \text{(by (3) in eq 8)}$$

$$= \sqrt{\frac{\pi}{2}}\, e^{-as}.$$

$$F_C\left(\frac{1}{a^2+x^2}\right) = \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{1}{a^2+x^2} \cos sx\, dx.$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{\pi}{2a} e^{-as}\right] \quad \text{(by (4) in Ex 8}$$

$$= \sqrt{\frac{\pi}{2}} \frac{1}{a} e^{-as}$$

Eg 13:- Find the F.S.T. of $\frac{1}{x}$.

Soln:

$$F_S\left(\frac{1}{x}\right) = \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{\sin sx}{x}\, dx.$$

Put $sx = \theta \Rightarrow$

$dx =$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{\sin\theta}{\theta}\, d\theta.$$

$$= \sqrt{2/\pi} \cdot \frac{\pi}{2}$$

[Contour Integral over semi circle

$$= \sqrt{\pi/2}.$$