TOPIC : 11 - **Convolution theorem**

Convolution Theorem.

Def:- Convolution of 2 functions.

The convolution of $f(x)$ and $g(x)$ is defined by $(f*g)(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)g(x-t)dt$.

Convolution Theorem:-

If $F\{f(x)\} = F(s)$ and $F\{g(x)\} = G(s)$, then

$F\{(f*g)(x)\} = F(s)G(s)$ where

$$(f*g)(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t).g(x-t)dt,$$ the

Convolution of $f(x)$ and $g(x)$.

Eg 14(B).

Verify convolution theorem for $f(x) = g(x) = e^{-x^2}$.

Soln:- Given $f(x) = g(x) = e^{-x^2}$

W.K.T. $F\{f(x)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x).e^{-isx} dx$

$$F\{e^{-x^2}\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-x^2}.e^{-isx} dx.$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-(x^2+isx)}\, dx.$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\left[\left(x-\frac{is}{2}\right)^2+\frac{s^2}{4}\right]} dx$$

$$= \frac{e^{-s^2/4}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\left(x-\frac{is}{2}\right)^2} dx.$$

Put $t = x - \frac{is}{2}$

$dt = dx$

$$= \frac{e^{-\frac{s^2}{4}}}{\sqrt{2\pi}}$$

$$= e^{-\frac{s^2}{4}}\left(\frac{1}{\sqrt{2\pi}}\right) \int_{-\infty}^{\infty} e^{-t^2} dt$$

$t^2 = u$

$2t\,dt = du$

$$= e^{-\frac{s^2}{4}}\left(\frac{1}{\sqrt{2\pi}}\right) \cdot 2 \int_{0}^{\infty} e^{-u} \frac{1}{2\sqrt{u}}\, du$$

$$= e^{-\frac{s^2}{4}}\left(\frac{1}{\sqrt{2\pi}}\right)\int_0^\infty e^{-u}u^{-1/2}\,du \qquad dt = \frac{du}{2t} = \frac{du}{2\sqrt{t}}$$

$$= e^{-\frac{s^2}{4}}\left(\frac{1}{\sqrt{2\pi}}\right)\cdot\sqrt{\pi}. \qquad \left[\because \int_0^\infty e^{-t}t^{-1/2}dt\right.$$

$$F\{f(x)\} = \frac{1}{\sqrt{2}}\cdot e^{-s^2/4}.$$

$$\therefore F\{g(x)\} = F\{f(x)\} = \frac{1}{\sqrt{2}}e^{-s^2/4}.$$

$$\therefore F\{f(x)\}\cdot F\{g(x)\} = \frac{1}{2}e^{-\frac{s^2}{2}} \longrightarrow (1).$$

$$f(x)*g(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(u)\,g(x-u)\,du.$$

$$e^{-x^2}*e^{-x^2} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-u^2}e^{-(x-u)^2}\,du.$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-(x-u)^2-u^2}\,du = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-x^2-u^2+2ux-u^2}\,du$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-x^2-2u^2+2ux}\,du = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-(x^2-2ux+2u^2)}\,du$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-2\left(\frac{x^2}{2}+u^2-ux+\left(\frac{x}{2}\right)^2-\left(\frac{x}{2}\right)^2\right)}\,du.$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-2\left[\frac{x^2}{2}+\left(u-\frac{x}{2}\right)^2-\frac{x^2}{4}\right]}\,du,$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-2\left[\left(u-x/2\right)^2+\frac{x^2}{4}\right]}\,du.$$

$$= e^{-x^2/2} \cdot \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-2\left(u-\frac{x}{2}\right)^2} du.$$

Put $t = u - \frac{x}{2}$

$dt = du$

$$= \frac{e^{-x^2/2}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-2t^2} dt$$

$$= \frac{e^{-x^2/2}}{\sqrt{2\pi}} \times 2 \int_{0}^{\infty} e^{-2t^2} dt.$$

Put $t^2 = y$

$2t\,dt = dy.$

$dt = \frac{dy}{2t}$

$= \frac{dy}{2\sqrt{y}}.$

$$= \frac{e^{-x^2/2}}{\sqrt{2\pi}} \cdot 2 \int_{0}^{\infty} e^{-2y} \frac{1}{2\sqrt{y}} dy$$

$$= \frac{e^{-x^2/2}}{\sqrt{2\pi}} \int_{0}^{\infty} e^{-2y}\, y^{-1/2} dy$$

$$= \frac{e^{-x^2/2}}{\sqrt{2\pi}} \cdot \frac{\sqrt{\pi}}{\sqrt{2}}.$$

$\left[\because \int_{0}^{\infty} e^{-2t}\, t^{-1/2} dt = \boxed{\frac{\sqrt{\pi}}{\sqrt{2}}}\right.$

$$= \frac{e^{-x^2/2}}{2}.$$

$$F(f(x) * g(x)) = F\left[\frac{1}{2} e^{-x^2/2}\right] = \frac{1}{2} F(e^{-x^2/2})$$

$$F(f(x) * g(x)) = \frac{1}{2} e^{-s^2/2} \longrightarrow (2).$$

$\therefore (1) = (2).$

Hence convolution theorem verified

Using Parsevals identity evaluate

$$\int_0^\infty \frac{dx}{(a^2+x^2)^2} \quad \text{(i)} \int_0^\infty \frac{x^2}{(a^2+x^2)^2}\,dx \quad \text{if } a>0.$$

Soln: W.K.T. if $f(x) = e^{-ax}$

then $F_s(f(x)) = \sqrt{\frac{2}{\pi}}\,\frac{s}{s^2+a^2} = F(s) \rightarrow ①$

$$F_c(f(x)) = \sqrt{\frac{2}{\pi}}\,\frac{a}{s^2+a^2} = F(s) \rightarrow ②$$

Parseval's identity is $\int_{-\infty}^{\infty} |f(x)|^2\,dx = \int_{-\infty}^{\infty} |F(s)|^2\,ds$

Here $f(x) = e^{-ax}$,

$$\int_{-\infty}^{\infty} |e^{-ax}|^2\,dx = \int_{-\infty}^{\infty}\left(\sqrt{\frac{2}{\pi}}\,\frac{s}{s^2+a^2}\right)^2 ds. \text{ using ①.}$$

$$2\int_0^\infty e^{-2ax}\,dx = \frac{2}{\pi}\int_{-\infty}^{\infty}\frac{s^2}{(s^2+a^2)^2}\,ds$$

$$2\left[\frac{e^{-2ax}}{-2a}\right]_0^\infty = \frac{2}{\pi}\cdot 2\int_0^\infty \frac{s^2}{(s^2+a^2)^2}\,ds.$$

$$\frac{1}{a} = \frac{4}{\pi}\int_0^\infty \frac{s^2}{(s^2+a^2)^2}\,ds$$

$$\int_0^\infty \frac{x^2}{(x^2+a^2)^2}\,dx = \frac{\pi}{4a}.$$

(ii). $\int_{-\infty}^{\infty} |f(x)|^2 dx = \int_{-\infty}^{\infty} |F(s)|^2 ds.$

Here $f(x) = e^{-ax}$.

$$\int_{-\infty}^{\infty} |e^{-ax}|^2 dx = \int_{-\infty}^{\infty} \left(\sqrt{\frac{2}{\pi}} \cdot \frac{a}{a^2+s^2}\right)^2 ds \text{ using } ②$$

$$\frac{1}{a} = \frac{2}{\pi} \cdot 2 \int_0^{\infty} \frac{a2}{(s^2+a2)^2} ds.$$

$$\int_0^{\infty} \frac{\cancel{a2}}{(s^2+a^2)^2} ds = \frac{\pi}{4a^3},$$