TOPIC : 10 - Transform of elementary functions

Example 9 :

Find the F.T. of $e^{-a|x|}$, $a>0$ and deduce that $\int_0^\infty \frac{\cos xt}{a^2+t^2}\,dt = \frac{\pi}{2a} e^{-a|x|}$. Hence find the F.T. of $x e^{-a|x|}$.

Soln:

The Fourier Transform of $f(x) = e^{-a|x|}$ is.

$$F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-a|x|} \cdot e^{isx}\,dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-a|x|} (\cos sx + i\sin sx)\,dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-a|x|} \cos sx\,dx + \frac{i}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-a|x|} \sin sx\,dx.$$

$$= \frac{2}{\sqrt{2\pi}} \int_0^\infty e^{-ax} \cos sx\,dx + 0 \quad \text{(odd function)}$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-ax} \cos sx\,dx = \sqrt{\frac{2}{\pi}} \left[\frac{e^{-ax}}{a^2+s^2} (-a\cos sx + s\sin sx) \right]_0^\infty$$

$$= \sqrt{\frac{2}{\pi}} \left(\frac{a}{a^2+s^2} \right).$$

By Fourier Inverse theorem,

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-isx} F(s)\, ds = \frac{1}{\sqrt{2\pi}} \cdot a\sqrt{\frac{2}{\pi}} \int_{-\infty}^{\infty} \frac{1}{s^2+a^2}(\cos sx - i \sin sx)\, ds$$

$$= \frac{a}{\pi} \int_{-\infty}^{\infty} \frac{\cos sx}{s^2+a^2}\, ds - \frac{ia}{\pi} \int_{-\infty}^{\infty} \frac{\sin sx}{s^2+a^2}\, ds.$$

$$= \frac{2a}{\pi} \int_{0}^{\infty} \frac{\cos sx}{s^2+a^2}\, ds - 0.$$

$$\therefore e^{-a|x|} = \frac{2a}{\pi} \int_{0}^{\infty} \frac{\cos sx}{s^2+a^2}\, ds \Rightarrow \int_{0}^{\infty} \frac{\cos sx}{s^2+a^2}\, ds = \frac{\pi}{2a} e^{-a|x|}$$

changing s to t, $\int_{0}^{\infty} \frac{\cos xt}{t^2+a^2}\, dt = \frac{\pi}{2a} e^{-a|x|}$.

To find $F(x e^{-a|x|})$.

By Property $F\{x f(x)\} = (-i) \frac{d}{ds} F(s)$ where

$F(s) = F\{f(x)\}$.

$f(x) = e^{-a|x|}$, we have.

$$F\{x e^{-a|x|}\} = \sqrt{\frac{2}{\pi}} (-i) \frac{d}{ds}\left(\frac{a}{s^2+a^2}\right)$$

$$= \sqrt{\frac{2}{\pi}} (-i) \left[\frac{(s^2+a^2)\cdot 0 - 2as}{(s^2+a^2)^2}\right].$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{2ias}{(s^2+a^2)^2}\right].$$

Example 12:

Find the Fourier Transform of $e^{-|x|}$ and hence find the F.T. of $e^{-|x|}\cos 2x$.

Soln:

$$F\{e^{-|x|}\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-|x|}\,[\cos sx + i\sin sx]\,dx.$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-|x|}\cos sx\,dx + 0.$$

$$= \sqrt{\frac{2}{\pi}}\int_{0}^{\infty} e^{-x}\cos sx\,dx.$$

$$= \sqrt{\frac{2}{\pi}}\left[\frac{e^{-x}}{1+s^2}(-\cos sx + s\sin sx)\right]_0^{\infty}$$

$$F\{e^{-|x|}\} = \sqrt{\frac{2}{\pi}}\left(\frac{1}{1+s^2}\right)$$

To find $F\{e^{-|x|}\cos 2x\}$.

By modulation theorem,

$$F\{f(x).\cos ax\} = \frac{1}{2}\left[F(s-a) + F(s+a)\right] \text{ where } F(s) = F\{f$$

$$F\{e^{-|x|}\cos 2x\} = \frac{1}{2}\sqrt{\frac{2}{\pi}}\left[\frac{1}{(s-2)^2+1^2} + \frac{1}{(s+2)^2+1^2}\right]$$

$$= \frac{1}{\sqrt{2\pi}}\left[\frac{1}{s^2-4s+5} + \frac{1}{s^2+4s+5}\right]$$

$$= \frac{1}{\sqrt{2\pi}}\left[\frac{s^2+5+4s+s^2-4s+5}{(s^2+5)^2-(4s)^2}\right]$$

$$= \frac{1}{\sqrt{2\pi}}\left[\frac{2(s^2+5)}{s^4-6s^2+25}\right].$$