Theorem

Prove that a connected graph G is Eulerian if and only if all the vertices are of even degree.

Proof

Let G be any graph having an Eulerian circuit and let C be an Eulerian circuit of G with origin (and terminus) vertex as u. Each time a vertex v occurs as an internal vertex of C, then two of the edges incident with v are accounted for degree.

We get, for internal vertex $v \in V(G)$

$$d(v) = 2 \times \left\{ \begin{array}{l} \text{number of times } v \text{ occur} \\ \text{inside the Euler circuit } C \end{array} \right.$$

$$= \text{even degree.}$$

and since an Euler circuit C contains every edge of G and C starts and ends at u.

$$\therefore d(u) = 2 + 2 \times \begin{cases} \text{number of times} \\ u \text{ occur inside } C \end{cases}$$

$$= \text{even degree}$$

$\therefore$ G has all the vertices of even degree.

Conversely, assume that each of its vertices has an even degree.

Claim: G has an Euler circuit.

Suppose not,

ii) Assuming G be a connected graph.

ll vertices of even degree and less number edges. i.e.) any graph having less no. of edges than G, then it has an Eulerian circuit.

Since each vertex of G has degree tleast two, therefore G contains closed path.

et C be a closed path of maximum possible ngth in G.

By assumption, C is not an Eulerian circu... G and G − E(C) has some component G′ ...ith |E(G′)| > 0. C has less no. of edges than ... therefore C itself is an Eulerian, and C ...as all the vertices of even degree, thus the ...nnected graph G′ also has all the vertices of ...ven degree.

Since |E(G′)| < |E(G)|, therefore G has ...n Euler circuit C′. Because G is connected, ...here is a vertex v in both C and C′. Now ...join C and C′ and transverse all the edges of C and C′ with common vertex v, we get CC′ is a closed path in G and |E(CC′)| > |E(C...

which is not possible for the choices of C.

∴ G has an Eulerian circuit.

∴ G is a Eluer Graph.

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