TOPIC : 9 – PARSEVAL'S THEOREM

Parseval's Theorem:

Let $f(x)$ be a periodic function with period 2π, defined in the interval $(-\pi, \pi)$. Then

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}[f(x)]^2\,d(x) = \frac{a_0^2}{4}+\frac{1}{2}\sum_{n=1}^{\infty}[a_n^2+b_n^2]$$

Root mean Square value:

The root mean square value of $f(x)$ over the interval (a,b) is defined as

$$R.M.S = \sqrt{\frac{\int_a^b [f(x)]^2 dx}{b-a}}$$

Problems:

① $f(x) = x^2$ in $(0,\pi)$.

Sol:

$$R.M.S = \sqrt{\frac{\int_a^b [f(x)]^2 dx}{b-a}} = \sqrt{\frac{\int_0^{\pi}(x^2)^2 dx}{\pi}}$$

$$= \sqrt{\frac{\left(\frac{x^5}{5}\right)_0^{\pi}}{\pi}} = \sqrt{\frac{\pi^5}{5\pi}} = \sqrt{\frac{\pi^4}{5}}$$

$$R.M.S = \frac{\pi^2}{\sqrt{5}}.$$

② $f(x) = 1-x$ in $0<x<1$.

Sol.

$$R.M.S = \sqrt{\frac{\int_a^b [f(x)]^2 dx}{b-a}} = \sqrt{\frac{\int_0^1 (1-x)^2 dx}{1}}$$

③ $f(x) = \pi - x$ in $0 < x < 2\pi$

Sol:

$$R.M.S = \sqrt{\frac{\int_a^b [f(x)]^2 dx}{b-a}} = \sqrt{\frac{\int_0^{2\pi} (\pi - x)^2 dx}{2\pi}}$$

$$= \sqrt{\frac{\left(\frac{(\pi - x)^3}{-3}\right)_0^{2\pi}}{2\pi}} = \sqrt{\frac{\frac{\pi^3}{3} + \frac{\pi^3}{3}}{2\pi}}$$

$$= \sqrt{\frac{2\pi^3}{3 \cdot 2\pi}} = \sqrt{\frac{\pi^2}{3}}$$

$$R.M.S = \frac{\pi}{\sqrt{3}}.$$

① Find the Fourier Series for $f(x) = x^2$ in $(-\pi, \pi)$

Hence deduce that $\frac{1}{1^4} + \frac{1}{2^4} + \cdots = \frac{\pi^4}{90}$

Sol: W.K.T,

$a_0 = \frac{2\pi^2}{3}$, $a_n = \frac{4}{n^2}(-1)^n$, $b_n = 0$.

Parseval's identity.

$$\frac{2}{b-a} \int_a^b [f(x)]^2 dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty} (a_n^2 + b_n^2)$$

$$\frac{2}{2\pi} \int_{-\pi}^{\pi} x^4 dx = \frac{\cancel{4}^2 \pi^4}{9 \cdot \cancel{2}} + \sum_{n=1}^{\infty} \frac{16}{n^4} (-1)^{2n}$$

$$\frac{1}{\cancel{16}}\left[\frac{2\pi^{\cancel{8}\,4}}{5} - \frac{2}{9}\pi^4\right] = \sum_{n=1}^{\infty} \frac{16}{n^4}$$

$$\frac{18\pi^4 - 10\pi^4}{45} = 16\sum_{n=1}^{\infty}\frac{1}{n^4}$$

$$\frac{8\pi^4}{45\times \cancel{16}_2} = \sum_{n=1}^{\infty}\frac{1}{n^4}$$

$$\frac{\pi^4}{90} = \sum_{n=1}^{\infty}\frac{1}{n^4}.$$

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(2) Find the half range cosine series of the function $f(x) = x(\pi - x)$ in $0 < x < \pi$. Hence deduce that $\frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \cdots = \frac{\pi^4}{90}$.

Sol:

W.K.T, $a_0 = \frac{\pi^2}{3}$, $a_n = \begin{cases} 0 & \text{if } n \text{ is odd} \\ \frac{-4}{n^2} & \text{if } n \text{ is even} \end{cases}$

$b_n = 0$.

By parseval's identity.

$$\frac{2}{b-a}\int_a^b [f(x)]^2\,dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty}(a_n^2 + b_n^2)$$

$$\frac{2}{\pi}\int_0^{\pi} (x(\pi - x))^2\,dx = \frac{\pi^4}{2.9} + \sum_{n=1}^{\infty}\frac{16}{n^4}$$

$$\frac{2}{\pi}\int_0^{\pi} x^2(\pi^2 + x^2 - 2x\pi)\,dx = \frac{\pi^4}{18} + \sum_{n=1}^{\infty}\frac{16}{n^4}$$

$$\frac{1}{\pi}\left(\frac{2\pi^{4}}{5}-\frac{2}{9}\pi^{4}\right)=\sum_{n=1}^{\infty}\frac{16}{n^4}$$

$$\frac{18\pi^4-10\pi^4}{45}=16\sum_{n=1}^{\infty}\frac{1}{n^4}$$

$$\frac{8\pi^4}{45\times 16}=\sum_{n=1}^{\infty}\frac{1}{n^4}$$

$$\frac{\pi^4}{90}=\sum_{n=1}^{\infty}\frac{1}{n^4}.$$

(2) Find the half range cosine series of the function $f(x)=x(\pi-x)$ in $0<x<\pi$. Hence deduce that $\frac{1}{1^4}+\frac{1}{2^4}+\frac{1}{3^4}+\cdots=\frac{\pi^4}{90}$.

Sol:

W.K.T, $a_0=\frac{\pi^2}{3}$, $a_n=\begin{cases}0 & \text{if } n \text{ is odd}\\ \frac{-4}{n^2} & \text{if } n \text{ is even}\end{cases}$

$b_n=0$.

By parseval's identity.

$$\frac{2}{b-a}\int_a^b [f(x)]^2\,dx=\frac{a_0^2}{2}+\sum_{n=1}^{\infty}(a_n^2+b_n^2)$$

$$\frac{2}{\pi}\int_0^{\pi}(x(\pi-x))^2\,dx=\frac{\pi^4}{2.9}+\sum_{n=1}^{\infty}\frac{16}{n^4}$$

$$\frac{2}{\pi}\int_0^{\pi}x^2(\pi^2+x^2-2x\pi)\,dx=\frac{\pi^4}{18}+\sum_{n=1}^{\infty}\frac{16}{n^4}$$

$$\frac{2}{l}\left(\frac{x^3}{3}\right)_0^l = \frac{l^2}{2} + \sum_{n=odd}^{\infty} \frac{16\, l^2}{n^4\pi^4}$$

$$\frac{2}{\cancel{l}}\cdot\frac{l^{3}}{3} = \frac{l^2}{2} + \frac{16l^2}{\pi^4}\sum_{n=odd}^{\infty}\left[\frac{1}{1^4}+\frac{1}{3^4}+\frac{1}{5^4}+\cdots\right]$$

$$\frac{2l^2}{3} - \frac{l^2}{2} = \frac{16l^2}{\pi^4}\left[\frac{1}{1^4}+\frac{1}{3^4}+\frac{1}{5^4}+\cdots\right]$$

$$\cancel{\frac{l^2}{6}} = \cancel{16 l}$$

$$\frac{\cancel{l^2}}{6}\,\frac{\pi^4}{16\cancel{l^2}} = \frac{1}{1^4}+\frac{1}{3^4}+\frac{1}{5^4}+\cdots$$

$$\frac{\pi^4}{96} = \frac{1}{1^4}+\frac{1}{3^4}+\frac{1}{5^4}+\cdots$$