TOPIC : 2 – FOURIER TRANSFORM

Fourier Transform pair.

The Fourier transform of $f(x)$ is defined by $F(f(x)) = F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{isx} dx$.

The inverse Fourier transform of $F(s)$ is defined by

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(s) e^{-isx} ds.$$

The Fourier transform of $F(s)$ of $f(x)$ and the inverse Fourier transform $f(x) = F^{-1}(F(s))$ are jointly called Fourier transform pair.

Parseval's theorem on Fourier Transform.

If $F(s)$ is the complex Fourier transform of $f(x)$, then.

$$\int_{-\infty}^{\infty} |f(x)|^2 dx = \int_{-\infty}^{\infty} |F(s)|^2 ds$$

Find the F.T. of $f(x) = \begin{cases} a-|x| & \text{for } |x|<a \\ 0 & \text{for } |x|>a>0. \end{cases}$

Deduce that (i) $\int_0^\infty \left(\frac{\sin t}{t}\right)^2 dt = \frac{\pi}{2}$, (ii) $\int_0^\infty \left(\frac{\sin t}{t}\right)^4 dt = \frac{\pi}{3}$

Soln:

$$F\{f(x)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{isx} dx.$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} (a-|x|) e^{isx} dx. = \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} (a-|x|)(\cos sx + i \sin sx) dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} (a-|x|) \cos sx \, dx$$

$$= \frac{2}{\sqrt{2\pi}} \int_{0}^{a} (a-|x|) \cos sx \, dx. = \sqrt{\frac{2}{\pi}} \int_0^a (a-x) \cos sx \, dx.$$

$$= \sqrt{\frac{2}{\pi}} \left[(a-x) \frac{\sin sx}{s} - (-1)\left(-\frac{\cos sx}{s^2}\right) \right]_0^a$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{1}{s^2} - \frac{\cos as}{s^2} \right]$$

$$F(s) = \sqrt{\frac{2}{\pi}} \frac{1}{s^2} (1 - \cos as)$$

By Fourier Inversion formula,

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(s) \cdot e^{-isx} ds$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \sqrt{\frac{2}{\pi}} \cdot \frac{1}{s^2} (1 - \cos as) \cdot e^{-isx} ds$$

$$= \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{1}{s^2} \left(2 \sin^2 \frac{as}{2}\right) e^{-isx} ds.$$

Show that the F.T. of $f(x) = \begin{cases} a^2 - x^2, & |x| < a \\ 0, & |x| > a > 0 \end{cases}$ is $2\sqrt{\frac{2}{\pi}} \left(\frac{\sin as - as \cos as}{s^3} \right)$. Hence deduce that (i) $\int_0^\infty \frac{\sin t - t \cos t}{t^3} dt = \frac{\pi}{4}$, (ii) $\int_0^\infty \left(\frac{\sin t - t \cos t}{t^3} \right)^2 dt = \frac{\pi}{15}$

Soln:

W.K.T. $F\{f(x)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{isx} dx.$

$$= \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} (a^2 - x^2) e^{isx} dx. = \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} (a^2 - x^2)(\cos sx + i \sin sx) dx$$

$$= \frac{1}{\sqrt{2\pi}} \left\{ \int_{-a}^{a} (a^2 - x^2) \cos sx \, dx + i \int_{-a}^{a} \cancelto{0}{(a^2 - x^2) \sin sx \, dx}. \right\}$$

$$= \frac{2}{\sqrt{2\pi}} \int_0^a (a^2 - x^2) \cos sx \, dx.$$

$$= \frac{2}{\sqrt{2\pi}} \left[(a^2 - x^2)\left(\frac{\sin sx}{s}\right) - (-2x)\left(\frac{-\cos sx}{s^2}\right) + (-2)\left(\frac{-\sin sx}{s^3}\right) \right]_0^a$$

$$= \frac{2}{\sqrt{2\pi}} \left[0 - \frac{2a \cos as}{s^2} + \frac{2 \sin as}{s^3} - (0 - 0 + 0) \right]$$

$$= \frac{4}{\sqrt{2\pi}} \left[\frac{\sin as - as \cos as}{s^3} \right]$$

$$F(s) = 2\sqrt{\frac{2}{\pi}} \left[\frac{\sin as - as \cos as}{s^3} \right]$$

(i) Using inverse F.T.,

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(s) \cdot e^{-isx} \, ds.$$

$$= \frac{1}{\sqrt{2\pi}} \, 2 \frac{\sqrt{2}}{\pi} \int_{-\infty}^{\infty} \frac{1}{s^3} (\sin as - as \cos as) \, e^{-isx} \, ds$$

$$= \frac{2}{\pi} \left\{ \int_{-\infty}^{\infty} \left(\frac{\sin as - as \cos as}{s^3} \right) \cos sx \, ds - \int_{-\infty}^{\infty} \left(\frac{\sin as - as \cos as}{s^3} \right) \underbrace{\sin sx \, ds}_{0} \right\}.$$

$$= \frac{2}{\pi} \int_{-\infty}^{\infty} \frac{\sin as - as \cos as}{s^3} \cos sx \, ds.$$

Put $a = 1$, we get.

$$1 - x^2 = \frac{4}{\pi}\int_0^\infty \frac{\sin s - s\cos s}{s^3}\cos sx\, ds.$$

$$= \frac{4}{\pi}\int_0^\infty \left(\frac{\sin t - t\cos t}{t^3}\right)\cos tx\, dt.$$

[$\because$ s is a dummy variable]

Put $x = 0$,

$$1 = \frac{4}{\pi}\int_0^\infty \frac{\sin t - t\cos t}{t^3}\, dt.$$

$$\boxed{\therefore \int_0^\infty \frac{\sin t - t\cos t}{t^3}\, dt = \frac{\pi}{4}}$$

(ii) Using Parseval's identity.

$$\int_{-\infty}^\infty |F(s)|^2\, ds = \int_{-\infty}^\infty |f(x)|^2\, dx$$

$$\frac{8}{\pi}\int_{-\infty}^\infty \left(\frac{\sin s - s\cos s}{s^3}\right)^2 ds. = \int_{-1}^1 (1 - x^2)^2\, dx.$$

$$= 2\int_0^1 (1 + x^4 - 2x^2)\, dx.$$

$$\frac{16}{\pi}\int_0^\infty \left(\frac{\sin s - s\cos s}{s^3}\right)^2 ds = \frac{16}{15}$$

$$\int_0^\infty \left(\frac{\sin s - s\cos s}{s^3}\right)^2 ds = \frac{\pi}{15}$$

$$\boxed{\therefore \int_0^\infty \left(\frac{\sin t - t\cos t}{t^3}\right)^2 dt = \frac{\pi}{15}}$$

[∵ s is a dummy variable]