TOPIC : 3 -Inverse Fourier transform

Example 4:

Find the Fourier transform of

$$f(x) = \begin{cases} 1, & |x| < 1 \\ 0, & |x| > 1 \end{cases}$$. Hence deduce that

(i) $\int_0^\infty \frac{\sin x}{x} dx = \frac{\pi}{2}$ (ii) $\int_0^\infty \frac{\sin^2 t}{t^2} dt = \frac{\pi}{2}$.

Soln:

By definition, F.T. of $f(x)$ is

$$F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{isx} f(x) dx.$$

$|x| < 1$
$\Rightarrow -1 < x < 1$

$$= \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} (\cos sx + i \sin sx) \cdot 1 \, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} \cos sx \, dx + \frac{i}{\sqrt{2\pi}} \int_{-1}^{1} \sin sx \, dx.$$

$$= \frac{2}{\sqrt{2\pi}} \int_0^1 \cos sx \, dx + \frac{i}{\sqrt{2\pi}} (0)$$

[by property of definite integral]

$$= \sqrt{\frac{2}{\pi}} \left[\frac{\sin sx}{s} \right]_0^1$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{\sin s}{s} \right]$$

(ii) By Inversion formula,

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-isx} F(s)\, ds$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \sqrt{\frac{2}{\pi}} \left(\frac{\sin s}{s}\right)(\cos sx - i \sin sx)\, ds$$

Example 3:- Find the F.T. of $f(x) = \begin{cases} 1 - x^2, & \text{if } |x| < 1 \\ 0, & \text{if } |x| > 1 \end{cases}$

Hence evaluate (i) $\int_0^\infty \frac{\sin t - t\cos t}{t^3}\, dt$ (ii) $\int_0^\infty \left(\frac{x\cos x - \sin x}{x^3}\right) \cos\frac{x}{2}\, dx$

Soln:

1 The Fourier transform of $f(x)$ is

$$F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{isx} f(x)\, dx = \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} (\cos sx + i \sin sx)(1 - x^2)\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} (1 - x^2) \cos sx\, dx + \frac{i}{\sqrt{2\pi}} \int_{-1}^{1} (1 - x^2) \sin sx\, dx \quad \underset{(\text{odd})}{} \quad 1 \to 0.$$

$$= \frac{2}{\sqrt{2\pi}} \int_0^1 (1 - x^2) \cos sx\, dx + 0 \quad (\text{even})$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} 2\sqrt{\frac{2}{\pi}} \left(\frac{\sin s - s\cos s}{s^3}\right)(\cos sx - i\sin sx)\, ds.$$

$$= \frac{2}{\pi} \int_{-\infty}^{\infty} \left(\frac{\sin s - s\cos s}{s^3}\right) \cos sx\, ds - \frac{2i}{\pi} \int_{-\infty}^{\infty} \left(\frac{\sin s - s\cos s}{s^3}\right) \sin sx\, ds$$

$$f(x) = \frac{4}{\pi} \int_0^{\infty} \left(\frac{\sin s - s\cos s}{s^3}\right) \cos sx\, ds. \quad \downarrow 0.$$

$$\therefore \int_0^{\infty} \left(\frac{\sin s - s\cos s}{s^3}\right) \cos sx\, dx = \frac{\pi}{4} f(x) \quad \text{②}$$

Put $x = 0$, in ②,

$$\int_0^{\infty} \left(\frac{\sin s - s\cos s}{s^3}\right) ds = \frac{\pi}{4} f(0) = 1$$

$$= \frac{\pi}{4}.$$

changing s to t, $\int_0^{\infty} \left(\frac{\sin t - t\cos t}{t^3}\right) dt = \frac{\pi}{4}.$

(ii). from ②, $\int_0^{\infty} \left(\frac{\sin s - s\cos s}{s^3}\right) \cos sx\, ds = \frac{\pi}{4} f(x)$

Put $x = \frac{1}{2}$, we have,

$$\int_0^\infty \left(\frac{\sin s - s\cos s}{s^3}\right) \cos\frac{s}{2}\, ds = \frac{\pi}{4}\left(\frac{3}{4}\right) = \frac{3\pi}{16}.$$

taking (−) sign

changing s to x, we get,

$$\int_0^\infty \left(\frac{x\cos x - \sin x}{x}\right) \cos\frac{x}{2}\, dx = \frac{-3\pi}{16}.$$