TOPIC:3- Groups

Groups

A non-empty set G together with the binary operation $*$, i.e) $(G, *)$ is called a group if $*$ satisfies the following conditions

(i) closure : $a * b \in G \quad \forall a, b \in G$

(ii) Associative : $(a * b) * c = a * (b * c), \ \forall a, b, c \in G$

(iii) Identity : $\exists$ an element $e \in G$ such that

$$a * e = e * a = a, \quad \forall a \in G.$$

then 'e' is called the identity element

(iv) Inverse : There exists an element $a^{-1} \in G$ called the inverse of 'a' such that

$$a * a^{-1} = a^{-1} * a = e, \quad \forall a \in G.$$

Abelian Group

In a group $(G, *)$, if $a * b = b * a$, $\forall a, b \in G$, then the group $(G, *)$ is called an abelian group.

Example $(Z, +)$ is an abelian group.

Problems based on Groups

1)

Prove that a group $(G, *)$ is abelian iff $(a * b)^2 = a^2 * b^2$, $\forall a, b \in G$.

Assume that G is abelian.

$\therefore \quad a * b = b * a \quad , \quad a, b \in G \rightarrow (1)$

$a^2 * b^2 = (a * a) * (b * b)$

$= a * [a * (b * b)]$ Associative

$= a * [(a * b) * b]$ Associative

$= a * [(b * a) * b]$ by (1)

$= (a * b) * (a * b)$ Associative

$a^2 * b^2 = (a * b)^2$

2)

Show that $(Q^+, *)$ is an abelian group where $*$ is defined by $a * b = \frac{ab}{2}$, $\forall a, b \in Q^+$.

Here Q^+ is the set of all positive rational number

(1) Closure

Clearly. $a * b = \frac{ab}{2} \in Q^+$

(2) Associative

$$(a * b) * c = \frac{ab}{2} * c = \frac{\frac{abc}{2}}{2}$$

$$= \frac{abc}{4}$$

$$a * (b * c) = a * \frac{bc}{2} = \frac{\frac{abc}{2}}{2}$$

$$= \frac{abc}{4}$$

$$\therefore (a * b) * c = a * (b * c) \quad \forall a, b, c \in Q^+$$

(3) Identity

Let 'e' be the identity element.

Then $a * e = a$

$$\Rightarrow \frac{ae}{2} = a \Rightarrow \boxed{e = 2}$$

$\therefore$ Identity element is $e = 2 \in Q^+$

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(4) Inverse Let a^{-1} be the inverse of 'a'.

Then $a * a^{-1} = e = 2$

$$\Rightarrow \frac{aa^{-1}}{2} = 2 \Rightarrow a^{-1} = \frac{4}{a}$$

$\therefore$ Inverse of a is $a^{-1} = \frac{4}{a} \in Q^+$.

(5) Commutative

Now $a * b = \frac{ab}{2}$

$$b * a = \frac{ba}{2} = \frac{ab}{2}$$

$\therefore a * b = b * a \quad \forall a, b \in Q^+$

$\therefore (Q^+, *)$ is an abelian group.

3)

Prove that the set $A = \{1, \omega, \omega^2\}$ is an abelian group of order 3 under usual multiplication where $1, \omega, \omega^2$ are cube roots of unity and $\omega^3 = 1$.

The following is the composition table of the elements in A with usual multiplication.

$\circ$	1	ω	ω^2
1	1	ω	ω^2
ω	ω	ω^2	1
ω^2	ω^2	1	ω

(1) Closure

All the elements in the above table are
the elements of A. Hence A is closed under

(2) Associative

Clearly multiplication of complex numbers are associative.

(3) Identity

The identity element is $1 \in A$.

(4) Inverse

$1 - 1$

$\omega - \omega^2$

$\omega^2 - \omega$.

(5) Commutative

From the table, we have

$$\omega \cdot \omega^2 = \omega^2 \cdot \omega = \omega^3 \text{ etc.}$$

Hence $(A, \cdot)$ is an abelian group.