



TOPIC: 5 - SOLUTION OF STANDARD TYPES OF FIRST ORDER PARTIAL DIFFERENTIAL EQUATIONS

Type : 3 $F(z, p, q) = 0$.

This eqn is of the form $f(z, p, q) = 0$ — (1)

Let $z = f(x+ay)$ be the solution of (1)

put $x+ay = u$ — (2)

Then $z = f(u)$ — (3)

Substitute $p = \frac{dz}{du}$ & $q = a \frac{dz}{du}$ Then

Integrating we get the solution.

1. Solve: $p(1+q) = qz$.

Sol:

Given $p(1+q) = qz$ — (1)

This eqn is of the form $f(z, p, q) = 0$

Let $u = x+ay$

$$\frac{\partial u}{\partial x} = 1 \quad \frac{\partial u}{\partial y} = a$$



$$p = \frac{dz}{du} \quad q = a \frac{dz}{du}$$

$$\textcircled{1} \Rightarrow \frac{dz}{du} \left(1 + a \frac{dz}{du} \right) = a z \frac{dz}{du}$$

$$\therefore 1 + a \frac{dz}{du} = a z$$

$$a \frac{dz}{du} = a z - 1$$

$$\frac{dz}{du} = \frac{a z - 1}{a}$$

$$\frac{du}{dz} = \frac{a}{a z - 1}$$

$$du = \frac{a}{a z - 1} dz$$

Integrating on both sides, we get

$$u = \log (a z - 1) + b$$

$$x + ay = \log (a z - 1)$$



② $z^2 = 1 + p^2 + q^2$.

Sol: Given $z^2 = 1 + p^2 + q^2$ — ①

This eqn is of the form $f(z, p, q) = 0$

Let $u = x + ay$

$$\frac{\partial u}{\partial x} = 1 \quad \frac{\partial u}{\partial y} = a$$
$$p = \frac{dz}{du} \quad q = a \frac{dz}{du}$$

① $\Rightarrow z^2 = 1 + \left(\frac{dz}{du}\right)^2 + a^2 \left(\frac{dz}{du}\right)^2$

$$\Rightarrow \left(\frac{dz}{du}\right)^2 (1 + a^2) = z^2 - 1$$
$$\Rightarrow \left(\frac{dz}{du}\right)^2 = \frac{z^2 - 1}{a^2 + 1}$$
$$\Rightarrow \frac{dz}{du} = \frac{\sqrt{z^2 - 1}}{\sqrt{a^2 + 1}}$$
$$\Rightarrow \sqrt{a^2 + 1} dz = \sqrt{z^2 - 1} du$$
$$\Rightarrow \sqrt{a^2 + 1} \frac{dz}{\sqrt{z^2 - 1}} = du$$

Integrating,

$$\sqrt{a^2 + 1} \int \frac{dz}{\sqrt{z^2 - 1}} = \int du + b$$

$\sqrt{a^2 + 1} \cosh^{-1}(z) = u + b$

$$\sqrt{a^2 + 1} \cosh^{-1}(z) = x + ay + b$$

is the complete solution.



③ $p + q = z.$

Sol:

Given $p + q = z \rightarrow \text{①}$

Let $u = x + ay$

$$p = \frac{dz}{du} \quad q = a \frac{dz}{du}$$

$$\frac{dz}{du} + a \frac{dz}{du} = z.$$

$$\frac{dz}{du} (1+a) = z$$

$$\frac{dz}{du} = \frac{z}{1+a}$$

$$(1+a) \frac{dz}{z} = du$$

Integrating,

$$(1+a) \int \frac{dz}{z} = \int du$$

$$(1+a) \log z = u + b$$

$$(1+a) \log z = x + ay + b. \text{ is the}$$

complete solution.



④ solve : $p(1-q^2) = q(1-z)$

Sol:

$$p(1-q^2) = q(1-z) \quad \text{--- (1)}$$

Let $u = x + ay$

$$p = \frac{dz}{du} \quad \& \quad q = a \frac{dz}{du}$$

$$\frac{dz}{du} \left(1 - a^2 \left(\frac{dz}{du} \right)^2 \right) = a \frac{dz}{du} (1-z)$$

$$1 - a^2 \left(\frac{dz}{du} \right)^2 = a \frac{dz}{du} (1-z)$$

$$a^2 \left(\frac{dz}{du} \right)^2 = -a + az + 1$$

$$a \frac{dz}{du} = \sqrt{1 - a + az}$$

$$\frac{a}{\sqrt{1 - a + az}} dz = du$$

$$a (1 - a + az)^{-\frac{1}{2}} dz = du$$

integrating we get

$$a \frac{(1 - a + az)^{\frac{1}{2}}}{\frac{1}{2} a} = u + b$$

$$2 (1 - a + az)^{\frac{1}{2}} = x + ay + b \text{ is the}$$

complete solution.



Type: (iv)

Equation containing x, y, p, q .

- i) Attach x & p in one side
- ii) Attach y & q in other side
- iii) Let it be equal to k
- iv) Find p & q
- v) $dz = p dx + q dy$
- vi) Integrate we get the complete solution.

① Solve: $p + q = x + y$ Sol:

Gn

$$p - x = y - q = k$$

$$p - x = k, \quad y - q = k$$

$$p = x + k, \quad q = y - k$$

$$dz = p dx + q dy$$

$$dz = (x + k) dx + (y - k) dy$$

Integrating,

$$z = \frac{x^2}{2} + kx + \frac{y^2}{2} - ky + b \text{ is the}$$

complete solution

Diff p.w.r. to b , $0 = 1$ is absurd

There is no singular solution

② Solve: $pq = xy$ Sol:

$$pq = xy$$



① Solve: $\frac{p}{x} = \frac{q}{y} = k$ (say)

$$\frac{p}{x} = k, \quad \frac{q}{y} = k.$$

$$p = xk, \quad q = \frac{y}{k}$$

$$dz = p dx + q dy \\ = (xk) dx + \left(\frac{y}{k}\right) dy$$

Integrating we get

$$z = \frac{x^2}{2} k + \frac{y^2}{2k} + b \text{ is the}$$

complete solution.

Diff p.w.r. to b , $0=1$ is absurd.

There is no singular integral.

⑧ Solve: $p^2 y (1+x^2) = q x^2$

Sol: Gn $p^2 y (1+x^2) = q x^2$

$$\Rightarrow \frac{p^2 (1+x^2)}{x^2} = \frac{q}{y} = k$$

$$\frac{p^2 (1+x^2)}{x^2} = k \quad \frac{q}{y} = k$$

$$p^2 = \frac{k x^2}{1+x^2} \quad q = yk.$$

$$p = \frac{\sqrt{k} x}{\sqrt{1+x^2}}$$

$$dz = p dx + q dy$$

$$dz = \frac{\sqrt{k} x}{\sqrt{1+x^2}} dx + yk dy$$

Integrate.

$$Z = \sqrt{k} \int \frac{x}{\sqrt{1+x^2}} dx + k \int y dy + b$$

$$Z = \sqrt{k} \sqrt{1+x^2} + k \frac{y^2}{2} + b \text{ is the complete}$$

Solution.

$$(4) \sqrt{p} + \sqrt{q} = x + y$$

Sol:

$$\sqrt{p} + \sqrt{q} = x + y$$

$$\sqrt{p} - x = y - \sqrt{q} = k$$

$$\sqrt{p} - x = k$$

$$y - \sqrt{q} = k$$

$$\sqrt{p} = x + k$$

$$\sqrt{q} = y - k$$

$$p = (x+k)^2$$

$$q = (y-k)^2$$

$$dz = p dx + q dy$$

$$dz = (x+k)^2 dx + (y-k)^2 dy$$

Integrate,

$$z = \frac{(x+k)^3}{3} + \frac{(y-k)^3}{3} + b \text{ is the}$$

complete solution.