SNS COLLEGE OF ENGINEERING

AN AUTONOMOUS INSTITUTION

Academic Year 2023 – 2024(Even semester)

19MA302 – TRANSFORMS & PARTIAL DIFFERENTIAL EQUATION

UNIT- II-FOURIER TRANSFORM
PART-A

1. **Prove that** $F_C[f(ax)] = \frac{1}{a}\mathrm{F_C}(\frac{s}{a})$ **,** where $\mathrm{F}(s) = F[f(x)]$

   [A.U./APR/MAY/2011]

   Answer:

$$F_c[f(x)] = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos sx\,dx$$

Let $u = a\,x$ then $x = u/a$ and hence $dx = \frac{du}{a}$

$$If \quad x = -\infty \quad then \quad u = -\infty$$

$$and \ if \quad x = \infty \quad then \quad u = \infty$$

$$F_C[f(ax)] = \sqrt{\frac{2}{\pi}}\int_0^\infty f(u)\cos(\frac{s}{a}u)\frac{du}{a}$$

$$= \frac{1}{a}\mathrm{F_C}(\frac{s}{a})$$

2. **Find Fourier Cosine Transform of** e^{-ax}, $\mathrm{x} > 0$

   [A.U./APR/MAY/2010]

   Answer:

   We know that,

$$F_c[f(x)] = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos sx\,dx$$

$$F_c[e^{-ax}] = \sqrt{\frac{2}{\pi}}\int_0^\infty e^{-ax}\cos sx\,dx$$

$$= \sqrt{\frac{2}{\pi}}\left[\frac{a}{s^2 + a^2}\right]$$

3. **Find Fourier Sine Transform of** e^{-ax}, x > 0

[A.U./APR/MAY/2010]

Answer:

We know that,

$$F_c[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \sin s\, x dx$$

$$F_c[e^{-ax}] = \rightleftarrows\rightleftarrows\rightleftarrows\rightleftarrows \sqrt{\frac{2}{\pi}} \int_0^{\infty} e^{-ax} \sin s\, x dx$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{s}{s^2+a^2}\right]$$

4. **Define self reciprocal with respect to Fourier transform.**

[A.U./NOV/DEC/2011]

Answer:

If the Fourier transform of *f(x)* is *f(s)* then *f(x)* is said to be self reciprocal function.

5. **State Convolution theorem in Fourier transform.**

[A.U./NOV/DEC/2012]

Answer:

$F[f(x) * g(x)] = F[f(x)]F[g(x)]$

6. **Prove that** $F[e^{iax} f(x)] = \mathrm{F}(s+a)$ where $\mathrm{F}(s) = F[f(x)]$

[A.U./NOV/DEC/2013]

Answer:

$$F[e^{iax} f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{iax} f(x) e^{isx} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{i(s+a)x} dx$$

$$= \mathrm{F}(s + a)$$

7. **Find Fourier Sine Transform of** $\frac{1}{x}$

[A.U./MAY/JUNE/2014] Answer:

$$F_s\left[\frac{1}{x}\right] = \sqrt{\frac{2}{\pi}} \int_0^{\infty} \frac{1}{x} \sin sx dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^{\infty} \frac{\sin sx}{x} dx$$

We know that,

$$\int_0^\infty \frac{\sin t}{t}\,dt = \frac{\pi}{2}$$

Let $sx = t \Rightarrow x = t/s \; and \; dx = dt/s$

$$F_s\left[\frac{1}{x}\right] = \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{\sin t}{t/s}\frac{dt}{s}$$

$$= \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{\not{s}\sin t}{t}\frac{dt}{\not{s}}$$

$$= \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{\sin t}{t}\,dt$$

$$= \sqrt{\frac{2}{\pi}}\frac{\pi}{2} = \sqrt{\frac{\pi}{2}}$$

UNIT – III

PARTIAL DIFFERENTIAL EQUATIONS

PART-A

1. **Form the PDE by eliminating the arbitrary constants a and b from the relation $z = ax^3+by^3$ [A.U. MAY/JUN 2014]**

Answer :

Given $z = ax^3 + by^3$ ... (1)

Partially differentiate with respect to x and y, we get

$p = \frac{\partial z}{\partial x} = 3ax^2$ and $q = \frac{\partial z}{\partial y} = 3by^2$

$\frac{p}{3} = ax^2 \Rightarrow \frac{px}{3} = ax^3$ ... (2)

$\frac{q}{3} = by^2 \Rightarrow \frac{qy}{3} = by^3$ ... (3)

Substitute (2) & (3) in (1), we get the required PDE is $3z = px + qy$.

2. **Form the PDE by eliminating the arbitrary constants a and b from the relatio ax^2+by^2 [A.U. NOV/DEC 2013]**

Answer :

Given $z = ax^2 + by^2$ ... (1)

Partially differentiate with respect to x and y, we get

$p = \frac{\partial z}{\partial x} = 2ax$ and $q = \frac{\partial z}{\partial y} = 2by$

$\frac{p}{2} = ax \quad \Rightarrow \quad \frac{px}{2} = ax^2$ ... (2)

$\frac{q}{2} = by^2 \quad \Rightarrow \quad \frac{qy}{2} = by^2$ ... (3)

Substitute (2) & (3) in (1), we get the required PDE is $2z = px + qy$.

3. Form the PDE from $(x\text{-}a)^2 + (y\text{-}b)^2 + z^2 = r^2$ **[A.U. MAY/JUN 2013]**

Answer:

Given $(x-a)^2 + (y-b)^2 + z^2 = r^2$ ...(1)

Partially differentiate with respect to x in (1), we get

$2(x-a) + 2z\frac{\partial z}{\partial x} = 0$, $(x-a) + zp = 0$ ...(2)

Partially differentiate with respect to y, we get

$2(y-b) + 2z\frac{\partial z}{\partial y} = 0$, $(y-b) + zq = 0$ ...(3)

(2) ⇒ x – a = - zp and

(3) ⇒ y – b = - zq

From (1), $(-zp)^2 + (-zq)^2 + z^2 = 1$

⇒ $z^2[p^2 + q^2 + 1] = 1$, which is the required PDE.

4. Find the complete integral of p+q = pq **[A.U. MAY/JUN 2013]**

Answer:

p + q = pq ...(1)

Let ax+by+c = 0 ...(2) be a solution.

Replace p by a & q by b in (1),

a + b = ab

⇒ a = ab - b

= b (a - 1)

⇒ $b = \frac{a}{a-1}$

From (2), $ax + \left(\frac{a}{a-1}\right)y + c = 0$, which is the complete solution.

5. Eliminate the arbitrary function f from z = f(y/x) and form the PDE.

Answer: **[A.U.NOV/DEC 2012]**

Given $z = f\left(\frac{y}{x}\right)$ ... (1)

Partially differentiate with respect to x in (1), we get

$p = \frac{\partial z}{\partial x} = f'\left(\frac{x}{y}\right)\left(\frac{-y}{x^2}\right)$

Partially differentiate with respect to y in (1), we get

$q = \frac{\partial z}{\partial y} = f'\left(\frac{x}{y}\right)\left(\frac{1}{x}\right)$

⇒ $\frac{p}{q} = \frac{-yx}{x^2}$

⇒ px + qy = 0, which is the required PDE.

6. Form PDE by eliminating the arbitrary function from z^2-xy = f(x/z)

Answer: **[A.U. MAY/JUN 2012]**

Given $z^2 - xy = f\left(\frac{x}{z}\right)$ ... (1)

Partially differentiate with respect to x, we get

$$2z\frac{\partial z}{\partial x} - y = f'\left(\frac{x}{z}\right)\left[\left(\frac{z - x\frac{\partial z}{\partial x}}{z^2}\right)\right]$$

$$\Rightarrow \quad 2zp - y = f'\left(\frac{x}{z}\right)\left[\left(\frac{z - xp}{z^2}\right)\right] \quad ...(2)$$

Partially differentiate with respect to y, we get

$$2z\frac{\partial z}{\partial y} - x = f'\left(\frac{x}{z}\right)\left[\left(\frac{0 - x\frac{\partial z}{\partial y}}{z^2}\right)\right]$$

$$\Rightarrow \quad 2zq - x = f'\left(\frac{x}{z}\right)\left[\left(\frac{-xq}{z^2}\right)\right] \quad ...(3)$$

$\frac{(2)}{(3)} \Rightarrow \frac{2zp - y}{2zq - x} = \frac{z - xp}{-xq} \Rightarrow x^2p + 2z^2q - xz - xyq = 0$, which is the required PDE.

7. Form PDE by eliminating the arbitrary constants a and b from $z = (x^2+a^2)(y^2+b^2)$

Answer: **[A.U. APR/MAY 2010]**

Given $z = (x^2 + a^2)(y^2 + b^2)$ ... (1)

Partially differentiate with respect to x in (1), we get

$p = \frac{\partial z}{\partial x} = 2x\,(y^2 + b^2)$

$i.e., \ (y^2 + b^2) = \frac{p}{2x}$

Partially differentiate with respect to y in (1), we get

$$q = \frac{\partial z}{\partial y} = 2y(x^2 + a^2)$$

$i.e., \ (x^2 + a^2) = \frac{q}{2y}$

$\therefore (1) \Rightarrow z = \left(\frac{q}{2y}\right)\left(\frac{p}{2x}\right)$

$\Rightarrow \quad 4xyz = pq$, which is the required PDE.

8. Form PDE by eliminating the arbitrary constants a and b from $z = (x^2+a)(y^2+b)$

Answer: **[A.U. APR/MAY 2011]**

Given $z = (x^2 + a)(y^2 + b)$ ... (1)

Partially differentiate with respect to x in (1), we get

$p = \frac{\partial z}{\partial x} = 2x\,(y^2 + b)$

$$i.e., \ (y^2 + b) = \frac{p}{2x}$$

Partially differentiate with respect to y in (1), we get $q = \frac{\partial z}{\partial y} = 2y(x^2 + a)$

$i.e., \ (x^2 + a) = \frac{q}{2y}$

$\therefore (1) \Rightarrow z = \left(\frac{q}{2y}\right)\left(\frac{p}{2x}\right)$

$\Rightarrow \quad 4xyz = pq$, which is the required PDE.

9. Find the partial differential equation of all planes cutting equal intercepts from the x and y axis . [A.U. NOV/DEC 2009]

Answer:

Equations of all planes cutting equal intercepts from the x and y axis are $\frac{x}{a}+\frac{y}{a}+\frac{z}{b}=1$

Partially differentiate with respect to x & y, we get

$\frac{1}{a}+\frac{p}{b}=0$ ...(1) *and* $\frac{1}{a}+\frac{q}{b}=0$ ...(2)

From (1) and (2), $\frac{p}{b}=\frac{q}{b} \Rightarrow p=q$

10. Find the complete integral of z = px + qy + $\sqrt{pq}$

Answer:

Given z = px + qy + $\sqrt{pq}$

This is in Clairaut's form (type 2)

Replace p by a and q by b, we get complete integral.

Hence z = ax + by + $\sqrt{ab}$

11. Find the complete integral of $\frac{z}{pq}=\frac{x}{q}+\frac{y}{p}+\sqrt{pq}$

Given $\frac{z}{pq}=\frac{x}{q}+\frac{y}{p}+\sqrt{pq}$

Multiply by pq we get

z = px + qy + $pq\sqrt{pq}$

This is in Clairaut's form (type 2)

Replace p by a and q by b, we get complete integral.

Hence z = ax + by + $ab\sqrt{ab}$

12. Find the complete integral of p – q = 0 [A.U.NOV/DEC 2008]

Answer:

Given p – q = 0 ...(1)

Let z = ax + by + c ...(2) be a solution

Replace p by a and q by b, we get

$a - b = 0 \quad \Rightarrow \quad b = a$

Hence, the complete integral is $z = ax + ay + c$

13. Find the solution of $\sqrt{p} + \sqrt{q} = 1$ **[A.U. NOV/DEC 2008]**

Answer:

Given $\sqrt{p} + \sqrt{q} = 1$ ...(1)

Let $z = ax + by + c$ ...(2) be a solution.

Replace p by a & q by b in (1), we get

$\sqrt{a} + \sqrt{b} = 1$

$\Rightarrow \sqrt{b} = 1 - \sqrt{a}$

$\Rightarrow b = (1 - \sqrt{a})^2$

Substitute b in (2), $z = ax + (1 - \sqrt{a})^2 y + c$ is the complete integral.

14. Form the partial differential equation by eliminating the arbitrary function from $\emptyset\left[z^2 - xy, \frac{x}{z}\right] = 0$ **[A.U. MAY/JUN 2006]**

Answer: Let $u = z^2 - xy$ and $v = \frac{x}{z}$

We know that,

$$\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} \\ \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y} \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} 2zp - y & \frac{z - px}{z^2} \\ 2zq - x & \frac{-xq}{z^2} \end{vmatrix} = 0$$

$$\Rightarrow px^2 - q\,(xy - 2z^2) = xz$$

15. Solve $(D - D')^3 z = 0$ **[A.U. NOV/DEC 2011]**

Answer:

The auxiliary equation is $(m - 1)^3 = 0$

i.e., $m = 1,1,1$

Therefore, the general solution is $z = f_1(y + x) + x f_2(y + x) + x^2 f_3(y + x)$

16. Solve $(D^4 - D'^4)Y = 0$ **[A.U. MAY/JUN 2014]**

Answer:

The auxiliary equation is $m^4 - 1 = 0$

Solving, $(m^2 - 1)(m^2 + 1) = 0$

$m^2 - 1 = 0$ and $m^2 + 1 = 0$

i.e., $m = 1, -1, i, -i.$

Therefore, the general solution is $z = f_1(y + x) + f_2(y - x) + f_3(y - ix) + f_4(y + ix)$

17. Solve $(D-1)(D-D'+1)z = 0$ **[A.U. NOV/DEC 2012]**

Answer:

The auxiliary equation is

$$(m-1)(m-1+1) = 0$$

Solving, $(m-1)m = 0$

i.e., $m = 1, 0$

Therefore, the general solution is $z = f_1(y + x) + f_2(y)$

18. Solve $(D^2-2DD'+D'^2)z = e^{x-y}$ **[A.U. NOV/DEC 2010]**

Answer:

The auxiliary equation is

$$m^2 - 2m + 1 = 0$$

Solving, $(m-1)(m-1) = 0$

i.e., $m = 1, 1$

The complementary function is $z = f_1(y + x) + x f_2(y + x)$

$$\text{P.I.} = \frac{1}{D^2-2DD'+D'^2} e^{x-y}$$

$$= \frac{e^{x-y}}{1-2(1)(-1)+1} = \frac{e^{x-y}}{4}$$

Hence, the general solution is $z = f_1(y + x) + x f_2(y + x) + \frac{e^{x-y}}{4}$

19. Solve $\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial x \partial Y} + \frac{\partial z}{\partial x} = 0$ **[A.U. Nov/Dec 2013]**

Answer :

Given equation can be written as $(D^2 - DD' + D)z = 0$

The auxiliary equation is

$$m^2 - m + m = 0$$

$$m^2 = 0$$

i.e., $m = 0, 0$

Therefore the general solution is $z = f_1(y) + x f_2(y)$

20. Solve $(D^2 - 7DD' + 6D'^2)z = 0$ **[A.U. MAY/JUN 2012]**

Answer:

The auxiliary equation is

$$m^2 - 7m + 6 = 0$$

Solving, $(m-1)(m-6) = 0$

i.e., $m = 1, 6$

The general solution is $z = f_1(y + x) + x f_2(y + 6x)$

21. Solve $(D^3 - 2D^2D')z = 0$ **[A.U. NOV/DEC 2009]**

Answer:

Given $(D^3 - 2D^2D')z = 0$

The auxiliary equation is

$m^3 - 2m^2 = 0$, Solving, $(m-2)m^2 = 0$

i.e., $m = 2, 0, 0.$

The general solution is $z = f_1(y + 2x) + f_2(y) + x f_3(y)$.

PART B (Unit II)

1. Find the Fourier cosine and sine transform of $f(x)=e^{-ax}$
2. Evaluate $\int_0^{\infty} \frac{dx}{(x^2+a^2)^2}$ using Parseval's identity
3. Using Fourier transform, evaluate $\int_0^{\infty} \frac{x^2 dx}{(x^2+a^2)^2}$
4. Evaluate $\int_0^{\infty} \frac{dx}{(x^2+a^2)(x^2+b^2)}$ using Fourier transform.
5. Evaluate $\int_0^{\infty} \frac{dx}{(x^2+1)(x^2+4)}$ using transform methods.
6. Find the Fourier sine and cosine transform of a function $f(x)=e^{-x}$. Using Parseval's identity, evaluate $\int_0^{\infty} \frac{dx}{(x^2+1)^2}$ and $\int_0^{\infty} \frac{x^2 dx}{(x^2+1)^2}$
7. Find the infinite Fourier transform of $e^{-a^2x^2}$. Hence deduce the infinite Fourier transform of $e^{x^2/2}$.
8. Find the Fourier Sine Transform $f(x) = \begin{cases} x, & 0 < x < 1 \\ 2-x, & 1 < x < 2 \\ 0, & x > 2 \end{cases}$

PART B(Unit III)

1. Form the pde by eliminating the arbitrary function from z = x $f\left(\frac{y}{x}\right) + y\varphi(x)$
2. Form the PDE by eliminating arbitrary function f and g from $z=x^2f(y)+y^2g(x)$
3. Find the singular integral of z = px + qy + $\sqrt{1+p^2+q^2}$
4. Solve z= px+qy+pq
5. Solve : $z = px + qy + p^2 - q^2$
6. Solve x(y – z)p + y(z – x)q = z(x – y)
7. Solve $x(y^2 - z^2)p + y(z^2 - x^2)q = z(x^2 - y^2)$
8. Solve the partial differential equation (mz–ny) p+(nx–lz) q=ly-mx
9. Solve $x(y^2+z)p+y(x^2+z)q = z(x^2-y^2)$
10. Solve the partial differential equation (3z–4y) p+(4x–2z) q=2y-3x

11. Solve $(D^3 - 7DD^{1^2} - 6D^{1^3})z = \cos(x+2y) + x + e^{x+2y}$

12. Solve $(D^3 - 7DD^{1^2} - 6D^{1^3})z = e^{2x+y} + \sin(x+2y)$

13. Solve $(D^3 + D^2D^1 - DD^{1^2} - D^{1^3})z = e^x \cos 2y$

14. Solve $(D^2 - 2DD^1)z = e^{2x-y} + x^3y$

15. Solve $(D^2 + 2DD' + D'^2)z = x^2y + e^{x-y}$

16. Solve$(D^2 + DD' - 6D'^2)z = y\cos x$

17. Solve $(D^2 - 5DD' + 6D'^2)z = y\sin x$