## Topic: 3. 9 – ENVELOPES

Problems - Envelope of two parameter family of curves.

1. Find the envelope of the family of lines $\frac{x}{a}+\frac{y}{b}=1$ where the parameters a and b are connected by the relation $a+b=c$.

Soln:

Given $\frac{x}{a}+\frac{y}{b}=1 \rightarrow (1)$

Taking differential we get.

$$-\frac{x}{a^2}da-\frac{y}{b^2}db=0 \rightarrow (2)$$

Given $a+b=c \rightarrow (3)$

Taking differential we get

$$da+db=0 \rightarrow (4)$$

$(2) \Rightarrow \frac{x}{a^2}da=-\frac{y}{b^2}db$

$(4) \Rightarrow da=-db$

$$\frac{(2)}{(4)} \Rightarrow \frac{\frac{x}{a^2}da}{da}=\frac{-\frac{y^2}{b^2}db}{-db}$$

$$\frac{x^2}{a^2}=\frac{y^2}{b^2}.$$

$$\frac{x/a}{a}=\frac{y/b}{b}=\frac{x/a+y/b}{a+b}=\frac{1}{c}$$

(i.e) $\frac{x}{a^2}=\frac{1}{c}$ ; $\frac{y}{b^2}=\frac{1}{c}$

$\Rightarrow a^2=xc$ ; $b^2=yc$.

$a = (xc)^{1/2}$ ; $b = (yc)^{1/2}$

③ $\Rightarrow$ $a + b = c$

$(xc)^{1/2} + (yc)^{1/2} = c$.

$\sqrt{x}\sqrt{c} + \sqrt{y}\sqrt{c} = c$

$\sqrt{c}\left[\sqrt{x} + \sqrt{y}\right] = c$

$\sqrt{x} + \sqrt{y} = \frac{c}{\sqrt{c}}$

$\sqrt{x} + \sqrt{y} = \sqrt{c}$.

2. Find the envelope of the family of lines $\frac{x}{a} + \frac{y}{b} = 1$ where the parameters a and b are connected by the relation $a^2 + b^2 = c^2$

Soln: Given $\frac{x}{a} + \frac{y}{b} = 1 \rightarrow ①$

Taking differentials; $-\frac{x}{a^2} da - \frac{y}{b^2} db = 0 \rightarrow ②$

Given $a^2 + b^2 = c^2 \rightarrow ③$

Taking differential $2ada + 2bdb = 0 \rightarrow ④$

② $\Rightarrow \frac{x}{a^2} da = -\frac{y}{b^2} db \rightarrow ⑤$

④ $\Rightarrow 2a\,da = -2b\,db \rightarrow ⑥$

⑤/⑥ $\Rightarrow \frac{x/a^2}{a} = \frac{y/b^2}{b} \Rightarrow \frac{x/a + y/b}{a^2 + b^2} = \frac{1}{c^2}$

$\frac{x}{a^3} = \frac{1}{c^2}$ ; $\frac{y}{b^3} = \frac{1}{c^2}$

$\Rightarrow a^3 = xc^2$ ; $b^3 = yc^2$

$a = (xc^2)^{1/3}$ $\quad b = (yc^2)^{1/3}$

Given $a^2 + b^2 = c^2$

$$(xc^2)^{2/3} + (c^2y)^{2/3} = c^2$$

$$(c^2)^{2/3}\left[x^{2/3} + y^{2/3}\right] = c^2$$

$$x^{2/3} + y^{2/3} = c^{2-4/3}$$

$$x^{2/3} + y^{2/3} = c^{2/3}$$

3. Find the envelope of $\frac{x}{l} + \frac{y}{m} = 1$ where $l$ and $m$ are connected by $\frac{l}{a} + \frac{m}{b} = 1$ and $a, b$ are constants.

Sol:

Given $\frac{x}{l} + \frac{y}{m} = 1 \rightarrow (1)$

Taking differentials.

$$-\frac{x}{l^2}dl - \frac{y}{m^2}dm = 0 \rightarrow (2)$$

Given $\frac{l}{a} + \frac{m}{b} = 1 \rightarrow (3)$

Taking differential $\frac{dl}{a} + \frac{dm}{b} = 0 \rightarrow (4)$

$$(2) \Rightarrow \frac{x}{l^2}\,dl = -\frac{y}{m^2}\,dm \rightarrow (5)$$

$$(4) \Rightarrow \frac{dl}{a} = -\frac{dm}{b} \rightarrow (6)$$

$$\frac{(5)}{(6)} \Rightarrow \frac{ax}{l^2} = \frac{-by}{m^2}$$

$$\frac{x/l}{l/a} = \frac{-y/m}{m/b} = \frac{x/l + y/m}{l/a + m/b} = \frac{1}{1} = 1$$

$$\frac{xa}{l^2} = 1 \quad ; \quad \frac{yb}{m^2} = 1$$

$$xa = l^2 \qquad yb = m^2$$

(i.e) $l = \sqrt{ax}$ ; $y = \sqrt{by}$

Given $\frac{l}{a} + \frac{m}{b} = 1$

$$\frac{\sqrt{ax}}{a} + \frac{\sqrt{ay}}{b} = 1$$

$$\sqrt{\frac{x}{a}} + \sqrt{\frac{y}{b}} = 1$$ which is the required envelope.