WHIRLING
OF
SHAFTS

- In the previous section, the rotor system —the shaft as well as the rotating body—was assumed to be rigid.

- However, in many practical applications, such as:
  - turbines,
  - compressors,
  - electric motors, and
  - pumps

  } heavy weight

  a heavy rotor is mounted on a lightweight, flexible shaft that is supported in bearings.

There will be unbalance in all rotors due to design and manufacturing errors.

Problems in shaft and a rotor systems:

i. Unbalance in rotor/disc
ii. Improper assembly
iii. Weaker bearings

- These unbalances as well as other effects, such as:

  - the stiffness and damping of the shaft,
  - gyroscopic effects, and
  - fluid friction in bearings,

will cause a shaft to bend in a complicated manner at certain rotational speeds, known as the whirling, whipping, or critical speeds.

Cont …

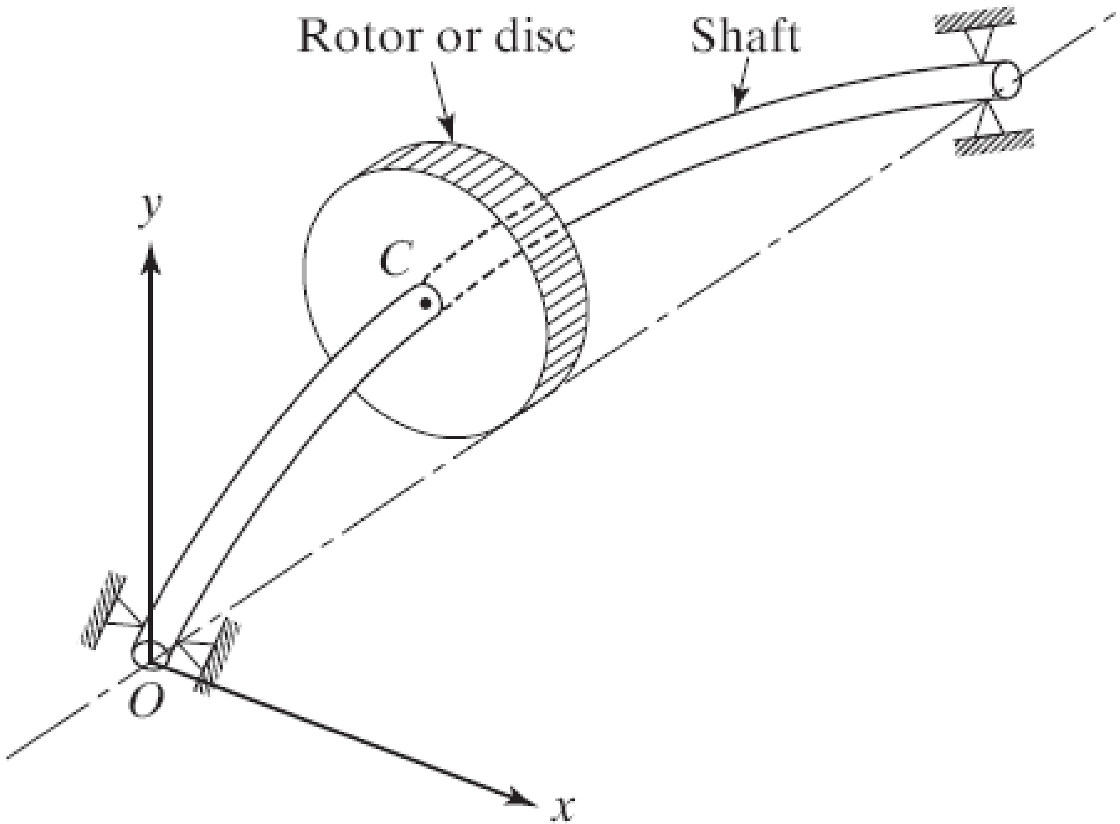
- Whirling is defined as the rotation of plane made by the bent shaft and line of centers of bearings.

- Whirling is also defined as *the rotation of the plane made by the line of centers of the bearings and the bent shaft.*

- *In this Chapter we consider the aspects of modeling the rotor system, critical speeds, response of the rotor system, and stability.*

Equations of Motion

- Consider a shaft supported by two bearings and carrying a rotor or disc of mass *m at the* middle:



- We shall assume that the rotor is subjected to a *steady-state excitation* due to mass unbalance.

- The forces acting on the rotor are:

  - the *inertia force* due to the acceleration of the mass center,
  - the *spring force* due to the elasticity of the shaft, and
  - the *external (Stationary)* & *internal (Rotary) damping forces*.

- The equations of motion of the rotor (mass *m) can be written as:*

$$\text{Inertia force } (F_i) = \text{Elastic force } (F_e) + \text{Internal damping force } (F_{di}) + \text{External damping force } (F_{de}) \quad \text{-- (6.1)}$$

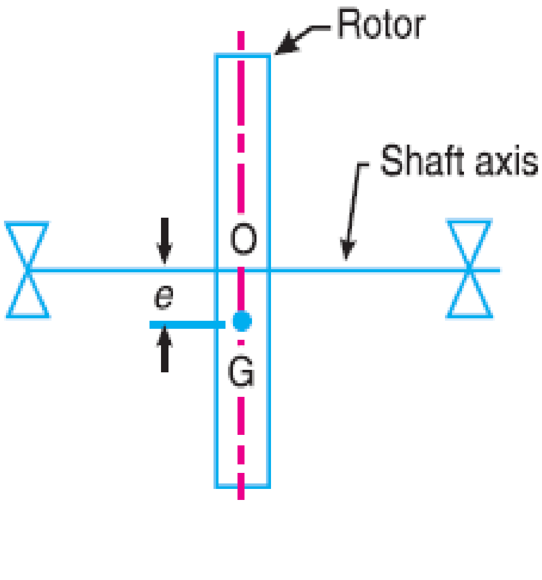
Critical or whirling speed of shaft

- In actual practice, a rotating shaft carries different mountings and accessories in the form of gears, pulleys, etc.
- The centre of gravity of the pulley or gear is at a certain distance from the axis of rotation due to this, the shaft is subjected to centrifugal force. This force will bent the shaft.
- The bending of shaft not only depends upon the value of eccentricity but also depends on the speed at which the shaft rotates.
- The speed at which the shaft runs so that the additional deflection of the shaft from the axis of rotation becomes infinite, is known as critical or whirling speed.

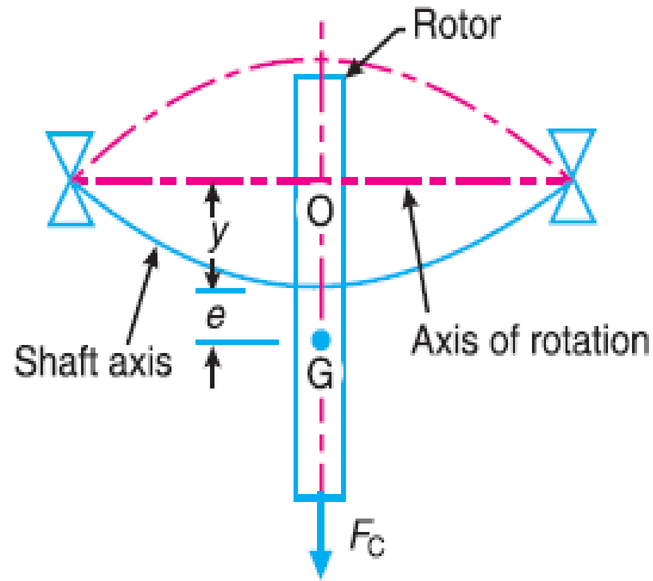
All rotating shafts, even in the absence of external load, will deflect during rotation.

- The unbalanced mass of the rotating object causes deflection that will create resonant vibration at certain speeds, known as the critical speeds.
- The magnitude of deflection depends upon the following:
  a) stiffness of the shaft and its support
  b) total mass of shaft and attached parts
  c) unbalance of the mass with respect to the axis of rotation
  d) the amount of damping in the system

Whirling Speed of a Shaft



(*a*) When shaft is stationary.



(*b*) When shaft is rotating.

m = Mass of the rotor,

e = Initial distance of centre of gravity of the rotor from the centre line of the bearing or shaft axis, when the shaft is stationary,

y = Additional deflection of centre of gravity of the rotor when the shaft starts rotating at ω rad/s, and

k = Stiffness of the shaft *i.e.* the load required per unit deflection of the shaft.

Since the shaft is rotating at ω rad/s, therefore centrifugal force acting radially outwards through G causing the shaft to deflect is given by:

$$F_C = m.\omega^2 (y + e)$$

- The shaft behaves like a spring. Therefore, the force resisting the deflection y,

$$= k.y$$

For the equilibrium position,

$$m\omega^2 (y+e) = k.y$$

$$y = \frac{m\omega^2 e}{k - m\omega^2} = \frac{\omega^2 e}{(k/m) - \omega^2} = \frac{\omega^2 e}{\omega_n^2 - \omega^2}$$

Since $\omega_n^2 = k/m$

A little consideration will show that when $\omega > \omega_n$, the value of y will be negative and the shaft deflects is the opposite direction as shown dotted

$$y = \pm \frac{\omega^2 e}{(\omega_n)^2 - \omega^2} = \frac{\pm e}{\left(\frac{\omega_n}{\omega}\right)^2 - 1} = \frac{\pm e}{\left(\frac{\omega_c}{\omega}\right)^2 - 1}$$

... (Substituting $\omega_n = \omega_c$)

- We see from the above expression that when $\omega_n = \omega_c$, the value of y becomes infinite. Therefore; ω_c is the critical or whirling speed.
- ∴ Critical or whirling speed

$$\omega_c = \omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{g}{\delta}} \text{ Hz}$$

Since F=kδ=mg

If N_c is the critical or whirling speed in r.p.s., then

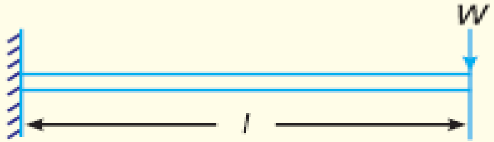
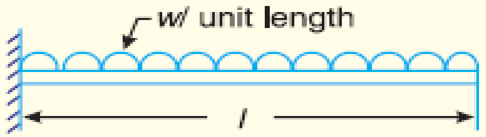
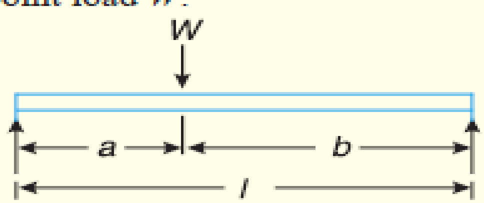
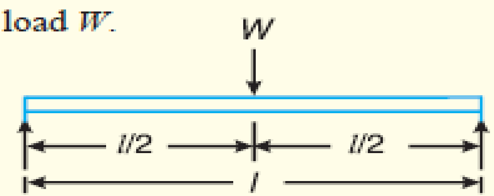
$$2\pi N_c = \sqrt{\frac{g}{\delta}} \quad \text{or} \quad N_c = \frac{1}{2\pi}\sqrt{\frac{g}{\delta}} = \frac{0.4985}{\sqrt{\delta}} \text{ r.p.s.}$$

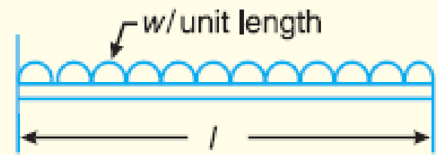
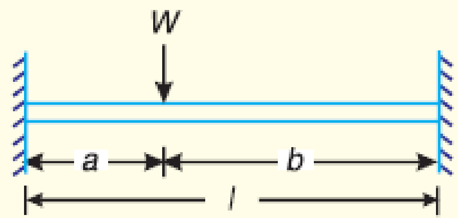
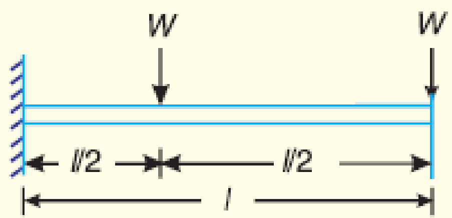
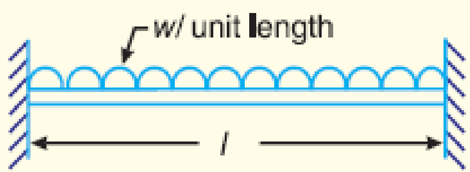
where

δ = Static deflection of the shaft in metres.

Hence the critical or whirling speed is the same as the natural frequency of transverse vibration but its unit will be revolutions per second.

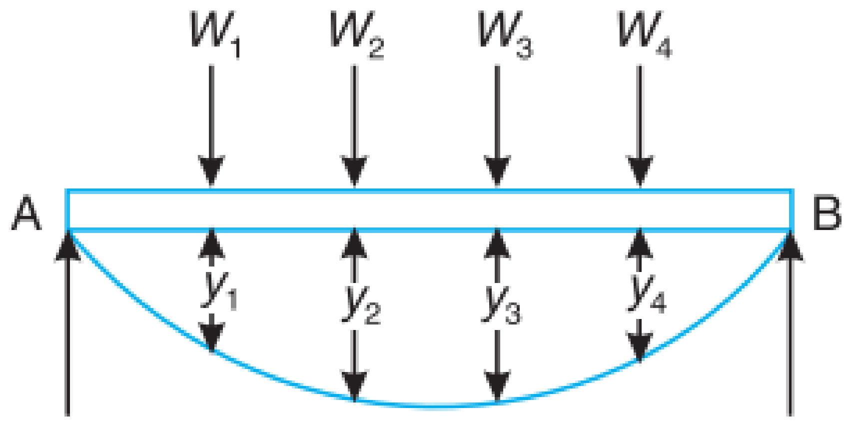
Values of static deflection (δ) for the various types of beams and under various load conditions.

S.No.	*Type of beam*	*Deflection (δ)*
1.	Cantilever beam with a point load W at the free end. 	$\delta = \frac{Wl^3}{3EI}$ (at the free end)
2.	Cantilever beam with a uniformly distributed load of w per unit length. 	$\delta = \frac{wl^4}{8EI}$ (at the free end)
3.	Simply supported beam with an eccentric point load W. 	$\delta = \frac{Wa^2b^2}{3EIl}$ (at the point load)
4.	Simply supported beam with a central point load W. 	$\delta = \frac{Wl^3}{48EI}$ (at the centre)

S.No.	*Type of beam*	*Deflection (δ)*
5.	Simply supported beam with a uniformly distributed load of *w* per unit length. 	$\delta = \frac{5}{384} \times \frac{wl^4}{EI}$ (at the centre)
6.	Fixed beam with an eccentric point load *W*. 	$\delta = \frac{Wa^3b^3}{3EIl}$ (at the point load)
7.	Fixed beam with a central point load *W*. 	$\delta = \frac{Wl^3}{192EI}$ (at the centre)
8.	Fixed beam with a uniformly distributed load of *w* per unit length. 	$\delta = \frac{wl^4}{384EI}$ (at the centre)

Natural Frequency of Free Transverse Vibrations For a Shaft Subjected to a Number of Point Loads

- Consider a shaft *AB* of negligible mass loaded with point loads W_1, W_2, W_3 and W_4 etc. in newton's, as shown in figure. Let m_1, m_2, m_3 and m_4 etc. be the corresponding masses in kg. The natural frequency of such a shaft may be found out by the following two methods: :



1. Energy (or Rayleigh's) method

- Let y_1, y_2, y_3, y_4 etc. be total deflection under loads W_1, W_2, W_3 and W_4 etc. as shown in figure. We know that maximum potential energy:

$$= \frac{1}{2} \times m_1 . g . y_1 + \frac{1}{2} \times m_2 . g . y_2 + \frac{1}{2} m_3 . g . y_3 + \frac{1}{2} \times m_4 . g . y_4 + ...$$

$$= \frac{1}{2} \Sigma m.g.y$$

- and maximum kinetic energy

$$= \frac{1}{2} \times m_1 (\omega . y_1)^2 + \frac{1}{2} \times m_2 (\omega . y_2)^2 + \frac{1}{2} \times m_3 (\omega . y_3)^2 + \frac{1}{2} \times m_4 (\omega . y_4)^2 + ...$$

$$= \frac{1}{2} \times \omega^2 \left[m_1 (y_1)^2 + m_2 (y_2)^2 + m_3 (y_3)^2 + m_4 (y_4)^2 + ... \right]$$

$$= \frac{1}{2} \times \omega^2 \Sigma m.y^2$$ . . . (where ω = Circular frequency of vibration)

Equating the maximum kinetic energy to the maximum potential energy, we have

$$\frac{1}{2} \times \omega^2 \, \Sigma m.y^2 = \frac{1}{2} \Sigma m.g.y$$

$$\therefore \qquad \omega^2 = \frac{\Sigma m.g.y}{\Sigma m.y^2} = \frac{g \, \Sigma m.y}{\Sigma m.y^2} \qquad \text{or} \qquad \omega = \sqrt{\frac{g \, \Sigma m.y}{\Sigma m.y^2}}$$

$\therefore$ Natural frequency of transverse vibration,

$$f_n = \frac{\omega}{2\pi} = \frac{1}{2\pi} \sqrt{\frac{g \, \Sigma m.y}{\Sigma m.y^2}}$$

2. *Dunkerley's method*

- The natural frequency of transverse vibration for a shaft carrying a number of point loads and uniformly distributed load is obtained from Dunkerley's empirical formula. According to this:

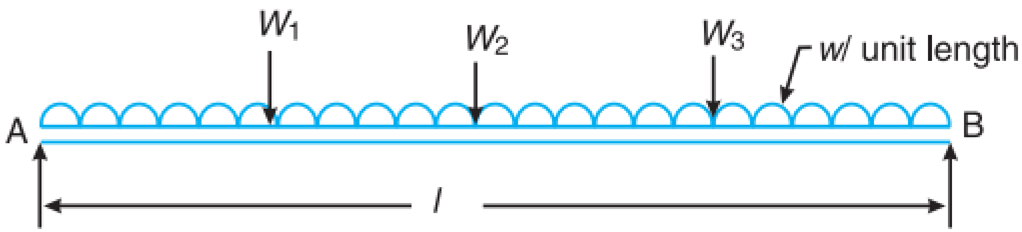
$$\frac{1}{(f_n)^2} = \frac{1}{(f_{n1})^2} + \frac{1}{(f_{n2})^2} + \frac{1}{(f_{n3})^2} + \dots + \frac{1}{(f_{ns})^2}$$

where f_n = Natural frequency of transverse vibration of the shaft carrying point loads and uniformly distributed load.

f_{n1}, f_{n2}, f_{n3}, etc. = Natural frequency of transverse vibration of each point load.

f_{ns} = Natural frequency of transverse vibration of the uniformly distributed load (or due to the mass of the shaft).

Now, consider a shaft AB loaded as shown in figure below:



Let $\delta_1, \delta_2, \delta_3,$ etc. = Static deflection due to the load W_1, W_2, W_3 etc. when considered separately.

δ_S = Static deflection due to the uniformly distributed load or due to the mass of the shaft.

We know that natural frequency of transverse vibration due to load W_1,

$$f_{n_1} = \frac{0.4985}{\sqrt{\delta_1}} \text{ Hz}$$

Similarly, natural frequency of transverse vibration due to load W_2,

$$f_{n2} = \frac{0.4985}{\sqrt{\delta_2}} \text{ Hz}$$

and, natural frequency of transverse vibration due to load W_3,

$$f_{n3} = \frac{0.4985}{\sqrt{\delta_3}} \text{ Hz}$$

Also natural frequency of transverse vibration due to uniformly distributed load or weight of the shaft,

$$f_{ns} = \frac{0.5615}{\sqrt{\delta_S}} \text{ Hz}$$

Therefore, according to Dunkerley's empirical formula, the natural frequency of the whole system,

$$\frac{1}{(f_n)^2} = \frac{1}{(f_{n1})^2} + \frac{1}{(f_{n2})^2} + \frac{1}{(f_{n3})^3} + \dots + \frac{1}{(f_{ns})^2}$$

$$= \frac{\delta_1}{(0.4985)^2} + \frac{\delta_2}{(0.4985)^2} + \frac{\delta_3}{(0.4985)^2} + \dots + \frac{\delta_S}{(0.5615)^2}$$

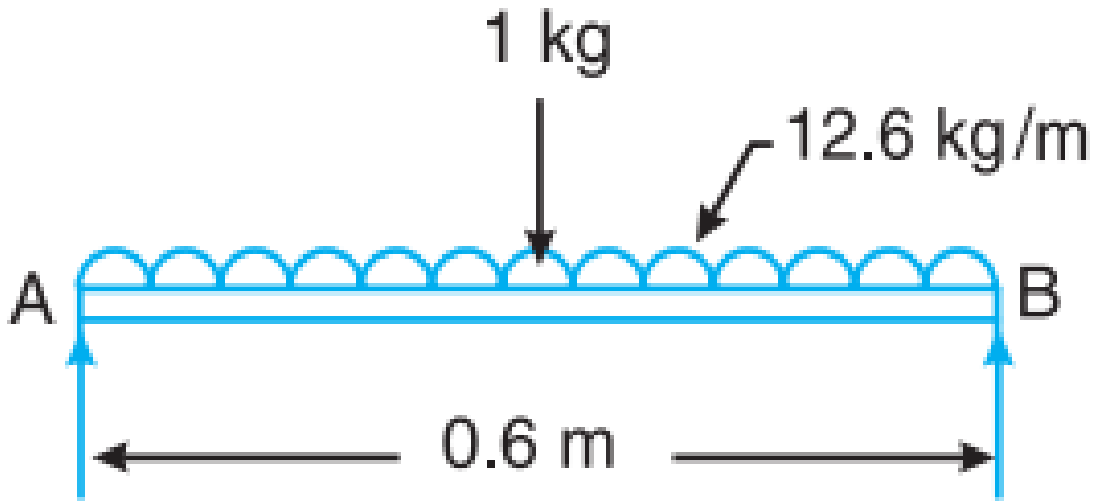
$$= \frac{1}{(0.4985)^2}\left[\delta_1 + \delta_2 + \delta_3 + \dots + \frac{\delta_S}{1.27}\right]$$

$$f_n = \frac{0.4985}{\sqrt{\delta_1 + \delta_2 + \delta_3 + \dots + \frac{\delta_S}{1.27}}} \text{ Hz}$$

Notes : 1. When there is no uniformly distributed load or mass of the shaft is negligible, then $\delta_S = 0$.

$$\therefore \quad f_n = \frac{0.4985}{\sqrt{\delta_1 + \delta_2 + \delta_3 + \dots}} \text{ Hz}$$

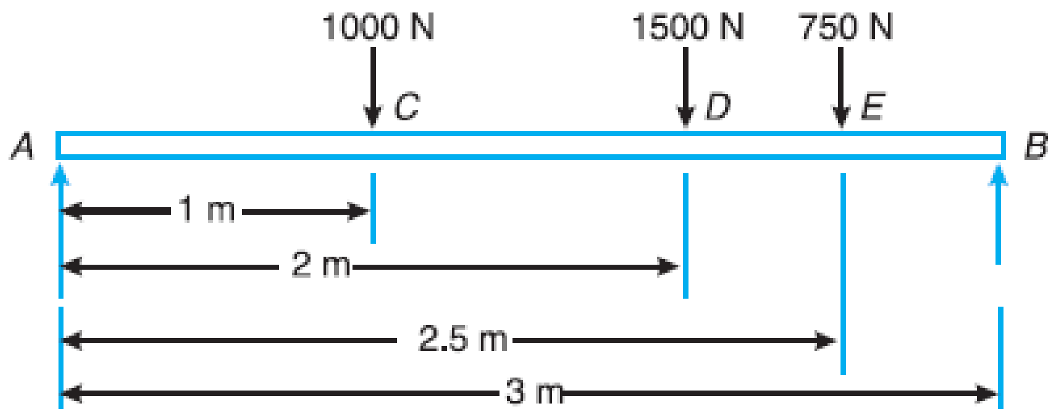
Example: Calculate the whirling speed of a shaft 20 mm diameter and 0.6 m long carrying a mass of 1 kg at its mid-point. The density of the shaft material is 40 Mg/m^3, and Young's modulus is 200 GN/m^2. Assume the shaft to be freely supported.



A shaft 50 mm diameter and 3 metres long is simply supported at the ends and carries three loads of 1000 N, 1500 N and 750 N at 1 m, 2 m and 2.5 m from the left support. The Young's modulus for shaft material is 200 GN/m2. Find the frequency of transverse vibration.

Solution. Given : $d = 50$ mm $= 0.05$ m ; $l = 3$ m, $W_1 = 1000$ N ; $W_2 = 1500$ N ; $W_3 = 750$ N; $E = 200$ GN/m^2 $= 200 \times 10^9$ N/m^2

The shaft carrying the loads is shown in Fig. 23.13



We know that moment of inertia of the shaft,

$$I = \frac{\pi}{64} \times d^4 = \frac{\pi}{64}(0.05)^4 = 0.307 \times 10^{-6}\,\text{m}^4$$

and the static deflection due to a point load W,

$$\delta = \frac{Wa^2b^2}{3EIl}$$

$\therefore$ Static deflection due to a load of 1000 N,

$$\delta_1 = \frac{1000 \times 1^2 \times 2^2}{3 \times 200 \times 10^9 \times 0.307 \times 10^{-6} \times 3} = 7.24 \times 10^{-3} \text{ m}$$

. . . (Here a = 1 m, and b = 2 m)

Similarly, static deflection due to a load of 1500 N,

$$\delta_2 = \frac{1500 \times 2^2 \times 1^2}{3 \times 200 \times 10^9 \times 0.307 \times 10^{-6} \times 3} = 10.86 \times 10^{-3} \text{ m}$$

. . . (Here a = 2 m, and b = 1 m)

and static deflection due to a load of 750 N,

$$\delta_3 = \frac{750 (2.5)^2 (0.5)^2}{3 \times 200 \times 10^9 \times 0.307 \times 10^{-6} \times 3} = 2.12 \times 10^{-3} \text{ m}$$

. . . (Here a = 2.5 m, and b = 0.5 m)

We know that frequency of transverse vibration,

$$f_n = \frac{0.4985}{\sqrt{\delta_1 + \delta_2 + \delta_3}} = \frac{0.4985}{\sqrt{7.24 \times 10^{-3} + 10.86 \times 10^{-3} + 2.12 \times 10^{-3}}}$$

$$= \frac{0.4985}{0.1422} = 3.5 \text{ Hz}$$ **Ans.**

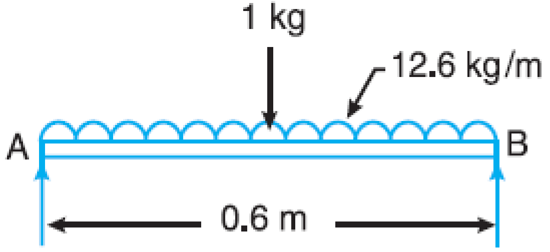
Ex. Calculate the whirling speed of a shaft 20 mm diameter and 0.6 m long carrying a mass of 1 kg at its mid-point. The density of the shaft material is 40 Mg/m³, and Young's modulus is 200 GN/m². Assume the shaft to be freely supported.

Solution. Given : d = 20 mm = 0.02 m ; l = 0.6 m ; m_1 = 1 kg ; ρ = 40 Mg/m^3 = 40×10^6 g/m^3 = 40×10^3 kg/m^3 ; E = 200 GN/m^2 = 200×10^9 N/m^2

We know that moment of inertia of the shaft,

$$I = \frac{\pi}{64} \times d^4 = \frac{\pi}{64}(0.02)^4 \text{ m}^4$$

$$= 7.855 \times 10^{-9} \text{ m}^4$$



Since the density of shaft material is 40×10^3 kg/m^3, therefore mass of the shaft per metre length,

$$m_S = \text{Area} \times \text{length} \times \text{density} = \frac{\pi}{4}(0.02)^2 \times 1 \times 40 \times 10^3 = 12.6 \text{ kg/m}$$

We know that static deflection due to 1 kg of mass at the centre,

$$\delta = \frac{Wl^3}{48\,EI} = \frac{1 \times 9.81(0.6)^3}{48 \times 200 \times 10^9 \times 7.855 \times 10^{-9}} = 28 \times 10^{-6} \text{ m}$$

and static deflection due to mass of the shaft,

$$\delta_S = \frac{5wl^4}{384EI} = \frac{5 \times 12.6 \times 9.81(0.6)^4}{384 \times 200 \times 10^9 \times 7.855 \times 10^{-9}} = 0.133 \times 10^{-3} \text{ m}$$

∴ Frequency of transverse vibration,

$$f_n = \frac{0.4985}{\sqrt{\delta + \frac{\delta_S}{1.27}}} + \frac{0.4985}{\sqrt{28 \times 10^{-6} + \frac{0.133 \times 10^{-3}}{1.27}}}$$

$$= \frac{0.4985}{11.52 \times 10^{-3}} = 43.3 \text{ Hz}$$

Let N_c = Whirling speed of a shaft.

We know that whirling speed of a shaft in r.p.s. is equal to the frequency of transverse vibration in Hz , therefore

$$N_c = 43.3 \text{ r.p.s.} = 43.3 \times 60 = 2598 \text{ r.p.m.}$$ **Ans.**

Example 23.6. *A shaft 1.5 m long, supported in flexible bearings at the ends carries two wheels each of 50 kg mass. One wheel is situated at the centre of the shaft and the other at a distance of 375 mm from the centre towards left. The shaft is hollow of external diameter 75 mm and internal diameter 40 mm. The density of the shaft material is 7700 kg/m³ and its modulus of elasticity is 200 GN/m². Find the lowest whirling speed of the shaft, taking into account the mass of the shaft.*

Solution. $l = 1.5$ m ; $m_1 = m_2 = 50$ kg ;

$d_1 = 75$ mm $= 0.075$ m ; $d_2 = 40$ mm $= 0.04$ m ;

$\rho = 7700$ kg/m^3 ; $E = 200$ GN/m^2 $= 200 \times 10^9$ N/m^2

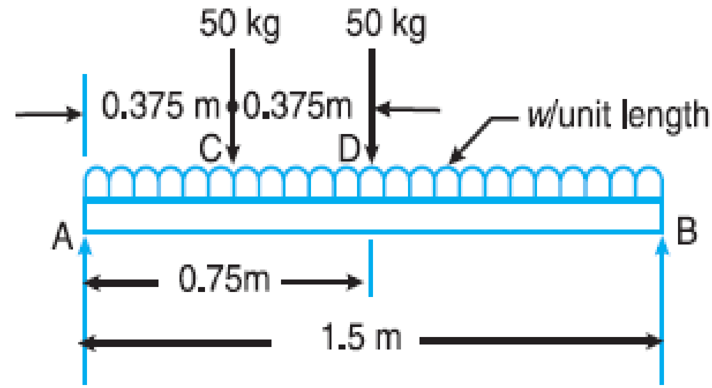


Fig. 23.16

moment of inertia of the shaft,

$$I = \frac{\pi}{64}\left[(d_1)^4 - (d_2)^4\right] = \frac{\pi}{64}\left[(0.075)^4 - (0.04)^4\right] = 1.4\times10^{-6}\,\text{m}^4$$

Since the density of shaft material is 7700 kg/m³, therefore mass of the shaft per metre length,

$$m_S = \text{Area} \times \text{length} \times \text{density} = \frac{\pi}{4}\left[(0.075)^2 - (0.04)^2\right]1\times7700 = 24.34\ \text{kg/m}$$

static deflection due to a load W

$$= \frac{Wa^2b^2}{3\,EIl} = \frac{m.ga^2b^2}{3\,EIl}$$

∴ Static deflection due to a mass of 50 kg at C,

$$\delta_1 = \frac{m_1 ga^2b^2}{3\,EIl} = \frac{50\times9.81(0.375)^2(1.125)^2}{3\times200\times10^9\times1.4\times10^{-6}\times1.5} = 70\times10^{-6}\ \text{m}$$

. . . (Here a = 0.375 m, and b = 1.125 m)

Similarly, static deflection due to a mass of 50 kg at *D*

$$\delta_2 = \frac{m_1 g a^2 b^2}{3\,EIl} = \frac{50 \times 9.81 (0.75)^2 (0.75)^2}{3 \times 200 \times 10^9 \times 1.4 \times 10^{-6} \times 1.5} = 123 \times 10^{-6} \text{ m}$$

. . . (Here $a = b = 0.75$ m)

static deflection due to uniformly distributed load or mass of the shaft,

$$\delta_S = \frac{5}{384} \times \frac{wl^4}{EI} = \frac{5}{384} \times \frac{24.34 \times 9.81 (1.5)^4}{200 \times 10^9 \times 1.4 \times 10^{-6}} = 56 \times 10^{-6} \text{ m}$$

. . . (Substituting, $w = m_S \times g$)

We know that frequency of transverse vibration,

$$f_n = \frac{0.4985}{\sqrt{\delta_1 + \delta_2 + \frac{\delta_S}{1.27}}} = \frac{0.4985}{\sqrt{70 \times 10^{-6} + 123 \times 10^{-6} + \frac{56 \times 10^{-6}}{1.27}}} \text{ Hz} = 32.4 \text{ Hz}$$

Since the whirling speed of shaft (N_c) in r.p.s. is equal to the frequency of transverse vibration in Hz, therefore

$$N_c = 32.4 \text{ r.p.s.} = 32.4 \times 60 = 1944 \text{ r.p.m.}$$ **Ans.**

Example 23.7. *A vertical shaft of 5 mm diameter is 200 mm long and is supported in long bearings at its ends. A disc of mass 50 kg is attached to the centre of the shaft. Neglecting any increase in stiffness due to the attachment of the disc to the shaft, find the critical speed of rotation and the maximum bending stress when the shaft is rotating at 75% of the critical speed. The centre of the disc is 0.25 mm from the geometric axis of the shaft. E = 200 GN/m².*

Solution. Given : $d = 5$ mm $= 0.005$ m ; $l = 200$ mm $= 0.2$ m ; $m = 50$ kg ; $e = 0.25$ mm $= 0.25 \times 10^{-3}$ m ; $E = 200$ GN/m^2 $= 200 \times 10^9$ N/m^2

Critical speed of rotation

moment of inertia of the shaft,

$$I = \frac{\pi}{64} \times d^4 = \frac{\pi}{64}(0.005)^4 = 30.7 \times 10^{-12}\,\text{m}^4$$

Since the shaft is supported in long bearings, it is assumed to be fixed at both ends. We know that the static deflection at the centre of the shaft due to a mass of 50 kg,

$$\delta = \frac{Wl^3}{192\,EI} = \frac{50 \times 9.81(0.2)^3}{192 \times 200 \times 10^9 \times 30.7 \times 10^{-12}} = 3.33 \times 10^{-3}\,\text{m} \quad \ldots (\because W = m.g)$$

We know that critical speed of rotation (or natural frequency of transverse vibrations),

$$N_c = \frac{0.4985}{\sqrt{3.33 \times 10^{-3}}} = 8.64 \text{ r.p.s.}$$ **Ans.**

Maximum bending stress

Let σ = Maximum bending stress in N/m², and

N = Speed of the shaft = 75% of critical speed = 0.75 N_c . . . (Given)

When the shaft starts rotating, the additional dynamic load (W_1) to which the shaft is subjected, may be obtained by using the bending equation,

$$\frac{M}{I} = \frac{\sigma}{y_1} \quad \text{or} \quad M = \frac{\sigma . I}{y_1}$$

We know that for a shaft fixed at both ends and carrying a point load (W_1) at the centre, the maximum bending moment

$$M = \frac{W_1 . l}{8}$$

$$\therefore \quad \frac{W_1 . l}{8} = \frac{\sigma \cdot I}{d/2} \qquad \ldots (\because \ y_1 = d/2)$$

$$W_1 = \frac{\sigma . I}{d/2} \times \frac{8}{l} = \frac{\sigma \times 30.7 \times 10^{-12}}{0.005/2} \times \frac{8}{0.2} = 0.49 \times 10^{-6} \sigma \text{N}$$

$\therefore$ Additional deflection due to load W_1,

$$y = \frac{W_1}{W} \times \delta = \frac{0.49 \times 10^{-6} \sigma}{50 \times 9.81} \times 3.33 \times 10^{-3} = 3.327 \times 10^{-12} \sigma$$

We know that

$$y = \frac{\pm e}{\left(\frac{\omega_c}{\omega}\right)^2 - 1} = \frac{\pm e}{\left(\frac{N_c}{N}\right) - 1}$$

. . . (Substituting $\omega_c = N_c$ and $\omega = N$)

$$3.327\times10^{-12}\sigma=\frac{\pm0.25\times10^{-3}}{\left(\frac{N_c}{0.75\,N_c}\right)^2-1}=\pm0.32\times10^{-3}$$

$$\sigma=0.32\times10^{-3}/3.327\times10^{-12}=0.0962\times10^{9}\,\mathrm{N/m^2}\quad\ldots(\text{Taking + ve sign})$$

$$=96.2\times10^{6}\ \mathrm{N/m^2}=96.2\ \mathrm{MN/m^2}$$ **Ans.**

Example 23.8. *A vertical steel shaft 15 mm diameter is held in long bearings 1 metre apart and carries at its middle a disc of mass 15 kg. The eccentricity of the centre of gravity of the disc from the centre of the rotor is 0.30 mm.*

*The modulus of elasticity for the shaft material is 200 GN/m^2 and the permissible stress is 70 MN/m^2. Determine : **1.** The critical speed of the shaft and **2.** The range of speed over which it is unsafe to run the shaft. Neglect the mass of the shaft.*

[For a shaft with fixed end carrying a concentrated load (W) at the centre assume $\delta = \dfrac{Wl^3}{192\,EI}$, *and* $M = \dfrac{W.l}{8}$, *where* δ *and M are maximum deflection and bending moment respectively].*

Solution. Given : d = 15 mm = 0.015 m ; l = 1 m ; m = 15 kg ; e = 0.3 mm = 0.3×10^{-3} m ; E = 200 GN/m^2 = 200×10^9 N/m^2 ; σ = 70 MN/m^2 = 70×10^6 N/m^2

moment of inertia of the shaft,

$$I = \frac{\pi}{64} \times d^4 = \frac{\pi}{64}(0.015)^4 = 2.5 \times 10^{-9}\,\text{m}^4$$

1. *Critical speed of the shaft*

Since the shaft is held in long bearings, therefore it is assumed to be fixed at both ends. We know that the static deflection at the centre of shaft,

$$\delta = \frac{Wl^3}{192\,EI} = \frac{15 \times 9.81 \times 1^3}{192 \times 200 \times 10^9 \times 2.5 \times 10^{-9}} = 1.5 \times 10^{-3}\,\text{m} \qquad \ldots(\because\ W = m.g)$$

$\therefore$ Natural frequency of transverse vibrations,

$$f_n = \frac{0.4985}{\sqrt{\delta}} = \frac{0.4985}{\sqrt{1.5 \times 10^{-3}}} = 12.88\,\text{Hz}$$

We know that the critical speed of the shaft in r.p.s. is equal to the natural frequency of transverse vibrations in Hz.

$\therefore$ Critical speed of the shaft,

N_c = 12.88 r.p.s. = 12.88 × 60 = 772.8 r.p.m. **Ans.**

2. *Range of speed*

Let N_1 and N_2 = Minimum and maximum speed respectively.

When the shaft starts rotating, the additional dynamic load ($W_1 = m_1.g$) to which the shaft is subjected may be obtained from the relation

$$\frac{M}{I} = \frac{\sigma}{y_1} \quad \text{or} \quad M = \frac{\sigma.I}{y_1}$$

Since $M = \frac{W_1.l}{8} = \frac{m_1.g.l}{8}$, and $y_1 = \frac{d}{2}$, therefore

$$\frac{m_1.g.l}{8} = \frac{\sigma.I}{d/2}$$

$$m_1 = \frac{8 \times 2 \times \sigma \times I}{d.g.l} = \frac{8 \times 2 \times 70 \times 10^6 \times 2.5 \times 10^{-9}}{0.015 \times 9.81 \times 1} = 19 \text{ kg}$$

$\therefore$ Additional deflection due to load $W_1 = m_1 g$,

$$y = \frac{W_1}{W} \times \delta = \frac{m_1}{m} \times \delta = \frac{19}{15} \times 1.5 \times 10^{-3} = 1.9 \times 10^{-3} \text{ m}$$

We know that,

$$y = \frac{\pm e}{\left(\frac{\omega_c}{\omega}\right)^2 - 1} \quad \text{or} \quad \pm \frac{y}{e} = \frac{1}{\left(\frac{N_c}{N}\right)^2 - 1}$$

. . . (Substituting, $\omega_c = N_c$, and $\omega = N$)

$$\therefore \quad \pm \frac{1.9 \times 10^{-3}}{0.3 \times 10^{-3}} = \frac{1}{\left(\frac{N_c}{N}\right)^2 - 1} \quad \text{or} \quad \left(\frac{N_c}{N}\right)^2 - 1 = \pm \frac{0.3}{1.9} = \pm 0.16$$

$$\left(\frac{N_c}{N}\right)^2 = 1 \pm 0.16 = 1.16 \text{ or } 0.84$$

. . . (Taking first plus sign and then negative sign)

$$N = \frac{N_c}{\sqrt{1.16}} \quad \text{or} \quad \frac{N_c}{\sqrt{0.84}}$$

$$\therefore \quad N_1 = \frac{N_c}{\sqrt{1.16}} = \frac{772.8}{\sqrt{1.16}} = 718 \text{ r.p.m.}$$

$$\text{and} \quad N_2 = \frac{N_c}{\sqrt{0.84}} = \frac{772.8}{\sqrt{0.84}} = 843 \text{ r.p.m.}$$

Hence the range of speed is from 718 r.p.m. to 843 r.p.m. **Ans.**

Summary

Due unbalance in a shaft-rotor system, rotating shafts tend to bend out at certain speed and whirl in an undesired manner.

The speed of the shaft under the condition when $r = 1$, *i.e* $\omega = \omega_n$ is referred as *critical speed of shaft.*

The theory developed helps the design engineer to select the speed of the shaft, which gives minimum deflection