Higher order partial differential equations

$$a \frac{\partial^2 z}{\partial x^2} + b \frac{\partial^2 z}{\partial y \partial x} + c \frac{\partial^2 z}{\partial y^2} = F(x,y)$$

Here a, b, c are constants

$F(x,y)$ = Functions of x & y.

Take $D = \frac{\partial}{\partial x}$, $D' = \frac{\partial}{\partial y}$

$$(aD^2 + bDD' + cD'^2)z = F(x,y) \rightarrow \text{①}$$

Solution of ① is given by.

$$z = C.F + P.I$$

To find C.F

Replace D by m

D' by 1

$\therefore a m^2 + bm + c = 0$

Solve we get two roots m_1, m_2.

case: ci)

$$\text{If } m_1 \neq m_2$$

$$C \cdot F = f_1(y + m_1 x) + f_2(y + m_2 x)$$

case: cii) If $m_1 = m_2$

$$C \cdot F = f_1(y + m_1 x) + x f_2(y + m_2 x)$$

To find P.I:

$$P.I = \frac{1}{aD^2 + bDD' + cD'^2} F(x, y)$$

Type: ci) (Homogeneous Equations)
R.H.S = 0.

① solve: $(D^2 - 5DD' + 6D'^2) z = 0$

Sol:

The A·E is $m^2 - 5m + 6 = 0$

$$(m-2)(m-3) = 0$$

$$\therefore m = 2, 3.$$

$$C \cdot F = f_1(y + 2x) + f_2(y + 3x)$$

$$P.I = 0.$$

$$\therefore z = C \cdot F + P.I$$

$$z = f_1(y + 2x) + f_2(y + 3x)$$

② solve: $(D^3 - 3DD'^2 + 2D'^3)$

Sol:

The A·E is $m^3 - 3m^2 + 2 = 0$.

$$
\begin{array}{c|ccc}
1 & 1 & 0 & -3 & 2 \\
 & \downarrow & 1 & 1 & -2 \\
\hline
1 & 1 & 1 & -2 & \boxed{0} \\
 & \downarrow & 1 & 2 & \\
\hline
-2 & 1 & 2 & \boxed{0} & \\
 & \downarrow & -2 & & \\
\hline
 & 1 & \boxed{0} & & \\
\end{array}
$$

$$\therefore \quad m = 1, \; 1, \; -2$$

C.F is $\quad f_1(y+x) + f_2(y+x) + f_3(y-2x)$

$$P.I = 0$$

$$Z = C.F + P.I$$

$$Z = f_1(y+x) + f_2(y+x) + f_3(y-2x)$$

Type: (ii)

$$R.H.S = e^{ax+by}$$

Working Rule:

① Replace D by a; D' by b

② If $dr \neq 0$ then we get P.I

③ If $dr = 0$ put and apply same method

dr w.r.to D and apply same method

① Solve: $(D^2 + 2DD' + D'^2) = e^{2x+3y}$ ⑭

Sol:

The A.E is $m^2 + 2m + 1 = 0$

$$(m+1)(m+1) = 0$$

$$m = -1, -1.$$

$$C.F = f_1(y-x) + x f_2(y-x)$$

$$P.I = \frac{1}{D^2 + 2DD' + D'^2} e^{2x+3y}$$

D by 2
D' by 3

$$= \frac{1}{(2)^2 + 2 \cdot 2 \cdot 3 + 3^2} e^{2x+3y}$$

$$= \frac{1}{4 + 12 + 9} e^{2x+3y}$$

$$= \frac{1}{25} e^{2x+3y}$$

$$z = C.F + P.I$$

$$z = f_1(y-x) + x f_2(y-x) + \frac{1}{25} e^{2x+3y}.$$

2. Solve: $(D^2 - D'^2) z = e^{x+2y}$

Sol:

The A.E is $m^2 - 1 = 0$

$$m^2 = 1$$

$$m = \pm 1.$$

$$P.I = \frac{1}{(D^2 - D'^2)} e^{x+2y}$$

$$P.I = \frac{-1}{3} e^{x+2y}$$

$$Z = C.F + P.I$$
$$= f_1(y-x) + f_2(y+x) - \frac{1}{3} e^{x+2y}$$