16 MARKS

EXAMPLE 2.2 A system vibrating with a natural frequency of 6 Hz starts with an initial amplitude (x_0) of 2 cm and an initial velocity ($\dot{x}_0$) of 25 cm/s. Determine the natural period, amplitude, maximum velocity, maximum acceleration and phase angle. Also write the equation of motion of a vibrating system.

Solution: **Given details:**

$$f = 6 \text{ Hz}$$
$$x_0 = 2 \text{ cm}$$
$$\dot{x}_0 = 25 \text{ cm/s}$$

The natural period is given by, $T = \frac{1}{f} = \frac{1}{6} = 0.167\text{s}$

The amplitude of motion $A = \sqrt{x_0^2 + \left(\frac{\dot{x}_0}{\omega_n}\right)^2}$

where, $\omega_n = 2\pi f = 2\pi(6) = 37.7 \text{ rad/s}$

$$A = \left[2^2 + \frac{25^2}{37.7^2}\right]^{1/2}$$

or $A = 2.11$ cm

The maximum velocity of a system is given by

$$\dot{x}_{max} = A\omega_n = 2.11 \times 37.7$$
$$= 79.44 \text{ cm/s}$$

The maximum acceleration of a system is $\ddot{x}_{max} = A\omega_n^2 = 2.11 \times 37.7^2$

$$= 2994.76 \text{ cm/s}^2$$

Phase angle

$$\phi = \tan^{-1}\left[\frac{x_0\omega_n}{\dot{x}_0}\right]$$
$$= \tan^{-1}\left[\frac{2 \times 37.7}{25}\right]$$
$$= 71°39'23''$$
$$= 1.25 \text{ rad.}$$

Equation of motion is $x = A\sin(\omega_n t + \phi) = 2.11\sin(37.7t + 1.25)$

EXAMPLE 2.3 A vertical cable 3 m long has a cross-sectional area of 4 cm² supports a weight of 50 kN. What will be the natural period and natural frequency of the system? $E = 2.1 \times 10^6$ kg/cm².

Solution: Given details:

$$A = 4 \text{ cm}^2$$
$$w = 50 \text{ kN}$$

∴ $$m = \frac{w}{g} = \frac{50 \times 10^3}{9.81} = 5096.8 \text{ kg}$$

$$E = 2.1 \times 10^6 \text{ kg/cm}^2$$

Stiffness $$k = \frac{AE}{L} = \frac{4 \times 2.1 \times 10^6}{300} = 7000 \text{ kg/cm}$$

$$= 7000 \times 981 = 6.867 \times 10^6 \text{ N/m}$$

Natural frequency, $$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{6.867 \times 10^6}{5096.8}}$$

$$= 36.7 \text{ rad/s}$$

Natural period, $$T = \frac{2\pi}{36.7}$$

$$= 0.17 \text{ s}$$

Frequency $$f = \frac{1}{T} = 5.84 \text{ Hz (or) cps}$$

EXAMPLE 2.4 A one kg mass is suspended by a spring having a stiffness of 1 N/mm. Determine the natural frequency and static deflection of the spring.

Solution: Given details:

$$k = 1 \text{ N/mm} = 1000 \text{ N/m}$$
$$m = 1 \text{ kg}$$

Natural frequency $$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{1000}{1}} = 31.62 \text{ rad/s}$$

$$f = \frac{\omega_n}{2\pi} = 5.03 \text{ Hz}$$

Static deflection δ_{st}

We know that, $$\omega_n = \sqrt{\frac{g}{\delta_{st}}}$$

$$\delta_{st} = \frac{g}{\omega_n^2} = 9.81 \times 10^{-3} \text{ m} = 9.81 \text{ mm}$$

EXAMPLE 2.5 A cantilever beam 3 m long supports a mass of 500 kg at its upper end. Find the natural period and natural frequency. $E = 2.1 \times 10^6$ kg/cm² and $I = 1300$ cm⁴.

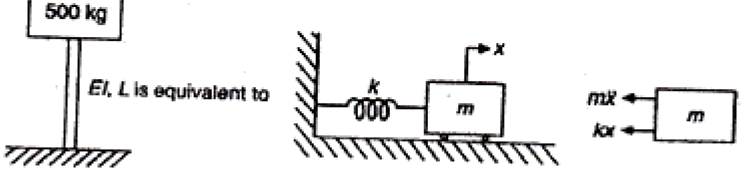

Figure 2.11

Solution:

Flexural stiffness for a cantilever beam, $k = \frac{3EI}{L^3}, k = \frac{3 \times 2.1 \times 10^6 \times 1300}{(300)^3}$

$= 303$ kg/cm $= 303 \times 981$ N/cm
$= 2.97 \times 10^5$ N/cm

Natural frequency $\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{2.97 \times 10^5}{500}}$

$= 24.37$ rad/s

or $f = \frac{\omega_n}{2\pi} = 3.88$ cps

Natural period $T = \frac{1}{f} = \frac{2\pi}{\omega_n} = 0.26$ s

EXAMPLE 2.6 A cantilever beam AB of length L is attached to a spring k and a mass M as shown in Figure 2.12. (a) Form the equation of motion; and (b) Find an expression for the frequency of motion.

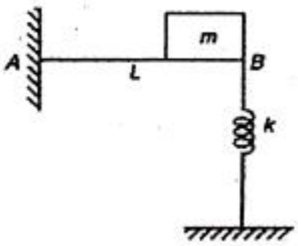

Figure 2.12

Solution:

(a) Equation of motion

Stiffness due to the applied mass M is $k_1 = \frac{M}{\Delta} = \frac{3EI}{L^3}$

This stiffness k_1 is acting parallel to k

$\therefore$ Equivalent spring stiffness $k_e = k_1 + k$

$$= \frac{3EI}{L^3} + k = \frac{3EI + kL^3}{L^3}$$

The differential equation of motion is

$$m\ddot{x} = -k_e x$$

$$\Rightarrow \qquad m\ddot{x} + k_e x = 0$$

$$\Rightarrow \qquad m\ddot{x} + \left[\frac{3EI + kL^3}{L^3}\right]x = 0$$

$$\Rightarrow \qquad \ddot{x} + \left[\frac{3EI + kL^3}{L^3 m}\right]x = 0$$

(b) The frequency of vibration is

$$f = \frac{1}{2\pi}\sqrt{\frac{k_e}{m}} = \frac{1}{2\pi}\sqrt{\frac{kL^3 + 3EI}{mL^3}}$$

EXAMPLE 2.7 Find the natural frequency of the system as shown in Figure 2.13. Take $k_1 = k_2 = 2000$ N/m, $k_3 = 3000$ N/m and $m = 10$ kg.

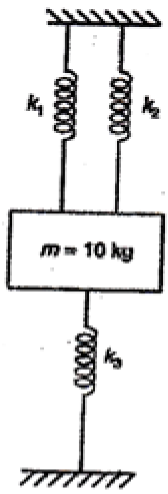

Figure 2.13

Solution: The equivalent system is shown in Figure 2.14.

Two springs k_1 and k_2 are in parallel

Equivalent stiffness $k_{e1} = k_1 + k_2 = 2000 + 2000$

$= 4000$ N/m

Again this equivalent spring is parallel to k_3

$\therefore$ Equivalent stiffness, $k_e = k_{e1} + k_3 = 4000 + 3000$

$= 7000$ N/m

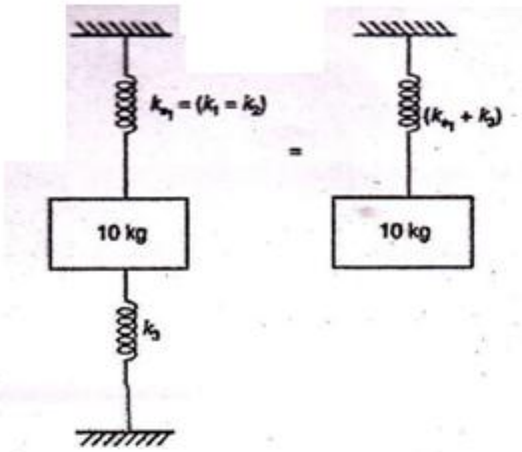

Figure 2.14

Natural frequency, $\omega_n = \sqrt{\frac{k_e}{m}} = \sqrt{\frac{7000}{10}}$

$= 26.46$ rad/s

or $f = \frac{\omega_n}{2\pi} = 4.21$ Hz

EXAMPLE 2.8 Consider the system shown in Figure 2.15.

If $k_1 = 2000$ N/m, $k_2 = 1500$ N/m, $k_3 = 3000$ N/m and $k_4 = k_5 = 500$ N/m, find the mass if the system has a natural frequency of 10 Hz.

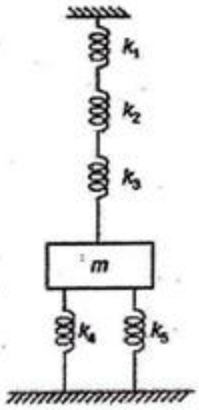

Figure 2.15

Solution: Given details:

$$k_1 = 2000 \text{ N/m}, \quad k_2 = 1500 \text{ N/m}$$
$$k_3 = 3000 \text{ N/m}, \quad k_4 = k_5 = 500 \text{ N/m}$$
$$f = 10 \text{ Hz.}$$

The springs k_1, k_2 and k_3 are in series. Their equivalent stiffness

$$\frac{1}{k_{e1}} = \frac{1}{k_1} + \frac{1}{k_2} + \frac{1}{k_3} = \frac{1}{2000} + \frac{1}{1500} + \frac{1}{3000}$$

or $$k_{e1} = 666.67 \text{ N/m.}$$

The two lower springs k_4 and k_5 are connected in parallel, so their equivalent stiffness

$$k_{e2} = k_4 + k_5 = 500 + 500 = 1000 \text{ N/m}$$

Again these two equivalent springs are in parallel,

$$k_e = k_{e1} + k_{e2} = 666.67 + 1000$$

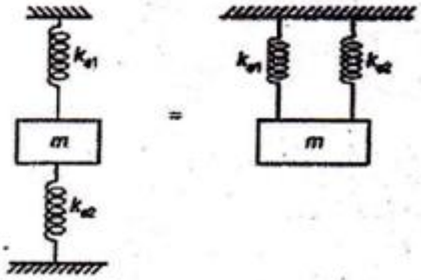

Figure 2.16 Equivalent spring.

⇒ $$k_e = 1666.67 \text{ N/m}$$

$$f = \frac{\omega_n}{2\pi}$$

⇒ $$\omega_n = 2\pi f = 2\pi(10)$$

⇒ $$\omega_n = 62.83 \text{ rad/s}$$

But $$\omega_n = \sqrt{\frac{k}{m}}$$

⇒ $$\omega_n^2 = \frac{k}{m}$$

⇒ $$m = \frac{k_e}{\omega_n^2} = \frac{1666.67}{(62.83)^2}$$

$$= 26.52 \text{ kg}$$

EXAMPLE 2.9 Determine the natural frequency of the system shown in Figure 2.17.

Solution:

For a cantilever beam, the stiffness is $k_b = \frac{3EI}{L^3} = \frac{3 \times 200 \times 10^9 \times 1.6 \times 10^{-5}}{4^3}$

$= 1.5 \times 10^5$ N/m

Now the beam and the spring k_1 are acting parallel. This combination is in series with k_2. This series combination is in parallel with k_3 and k_4.

$$\therefore \quad k_{e1} = k_3 + k_4 = 4 \times 10^5 + 6 \times 10^5$$

$$= 10 \times 10^5 \text{ N/m}$$

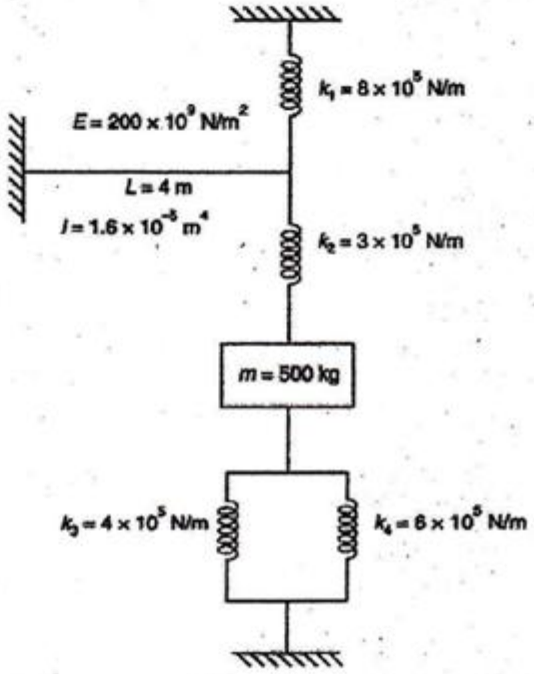

Figure 2.17

But $$k_e = \frac{1}{\left(\frac{1}{k_b + k_1}\right) + \frac{1}{k_2}} + k_{e1} = \frac{1}{\left(\frac{1}{1.5 \times 10^5 + 8 \times 10^5}\right) + \frac{1}{3 \times 10^5}} + (10 \times 10^5)$$

$\Rightarrow \quad k_e = 12.28 \times 10^5$ N/m.

Natural frequency $\omega_n = \sqrt{\frac{k_e}{m}} = \sqrt{\frac{12.28 \times 10^5}{500}}$

$= 49.56$ rad/s

EXAMPLE 2.10 A simply supported rectangular beam has a span of 1 m. It is 100 mm wide and 10 mm deep. It is connected at mid-span of a beam by means of a linear spring having a stiffness of 100 kg/cm and a mass of 300 kg is attached at the other end of spring. Determine the natural frequency of the system. Take $E = 2.1 \times 10^6$ kg/cm^2.

Solution:

The stiffness of simply supported beam is $k_b = \frac{48EI}{L^3}$

$$I = \frac{bd^3}{12} = \frac{10(1)^3}{12} = 0.833 \text{ cm}^4$$

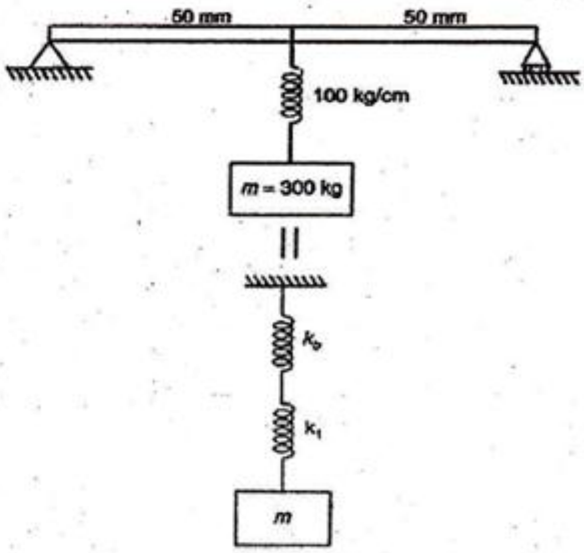

Figure 2.18

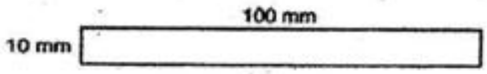

Figure 2.19

$$k_b = \frac{48 \times 2.1 \times 10^6 \times 0.833}{(100)^3}$$

$$= 84 \text{ kg/cm}$$

The two springs k_1 and k_2 are in series, the equivalent stiffness of the compound spring is given by

$$\frac{1}{k_e} = \frac{1}{k_b} + \frac{1}{k_1} = \frac{1}{84} + \frac{1}{100}$$

$\Rightarrow$ $k_e = 45.65$ kg/cm

$$= 45.65 \times 981 = 0.448 \times 10^5 \text{ N/cm}$$

Natural frequency $\omega_n = \sqrt{\frac{k_e}{m}} = \sqrt{\frac{0.448 \times 10^5}{300}}$

$$= 12.22 \text{ rad/s}$$

Logarithmic Decrement Method

This method is used to measure damping in time domain. In this method, the free vibration displacement amplitude history of a system is measured and recorded.

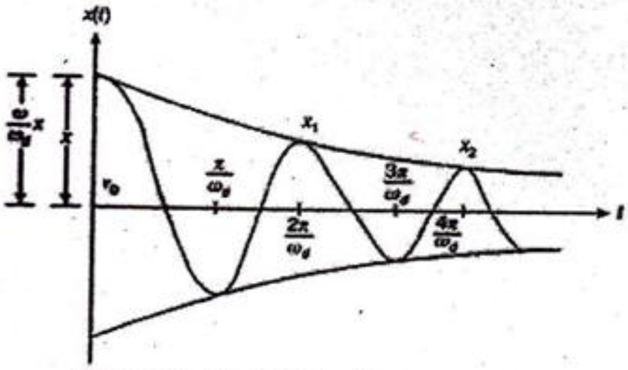

Figure 3.11 Amplitude decay for viscous damping.

shows the amplitude decay curve. Logarithmic decrement is defined as the natural logarithmic value of the ratio of two adjacent peak values of displacement in free vibration. It is a dimensionless parameter. It is denoted by a symbol δ.

$$\delta = \ln\frac{x_1}{x_2}$$

In case of underdamped system, the general solution is given by

$$x = Xe^{-\rho\omega_n t}(\sin\omega_d t + \phi)$$

Let the displacement within one cycle is x_0.

$$\text{Amplitude } x_0 = xe^{-\rho\omega_n t}$$

Let x_1 is the displacement after one cycle

$$\text{Amplitude } x_1 = xe^{-\rho\omega_n}(t + T_d)$$

⇒
$$\frac{x_0}{x_1} = \frac{xe^{-\rho\omega_n t}}{xe^{-\rho\omega_n}(t+T_d)}$$

$$= e^{-\rho\omega_n T_d}$$

Taking logarithm on both sides

$$\log\frac{x_0}{x_1} = \rho\omega_n T_d$$

$$\delta = \rho\omega_n T_d = \frac{c}{c_c}\omega_n T_d = \frac{c}{2m\omega_n}\omega_n T_d$$

⇒
$$\delta = \frac{c}{2m}T_d$$

where
$$T_d = \frac{2\pi}{\omega_n\sqrt{1-\rho^2}}$$

⇒
$$\delta = \frac{c}{2m}\frac{2\pi}{\omega_n\sqrt{1-\rho^2}} = \frac{c}{c_c}\frac{2\pi}{\sqrt{1-\rho^2}}$$

⇒
$$\delta = \frac{2\pi\rho}{\sqrt{1-\rho^2}}$$