TOPIC:6- Lattices as algebraic systems

Lattice:

A lattice is a partially ordered set $(L, \leq)$ in which every pair of elements $a, b \in L$ has a greatest lower bound and a least upper bound.

Theorem:

Let $(L, \leq)$ be a lattice in which $*$ and $\oplus$ denotes the operations of meet and join respectively. For any $a, b \in L$

$$a \leq b \Leftrightarrow a * b = a \Leftrightarrow a \oplus b = b.$$

Proof:

First let us prove that $a \leq b \Leftrightarrow a * b = a$

Let us assume that $a \leq b$

and also we know that $\cancel{a \leq b}$ $a \leq a$.

$\therefore a \leq a * b$ — (1) $[\because a \leq b \;\&\; a \leq c \Rightarrow a \leq b * c]$

Since $a * b$ is the GLB of $a \& b$

we have $a * b \leq a$ — (2)

Hence $a \leq b \Rightarrow a * b = a$ [from ① & ②]

Conversely we assume that $a * b = a$, but it is only possible if $a \leq b$

(ie) $a * b = a \Rightarrow a \leq b$

Combining these two results, we get

$a \leq b \Leftrightarrow a * b = a$.

We next prove $a * b = a \Leftrightarrow a \oplus b = b$

Suppose $a * b = a$, then

$b = b \oplus (a * b)$ [by absorption law]

$= b \oplus a$

$\Rightarrow a \oplus b = b$

Conversely assume that $a \oplus b = b$

$a = a * (a \oplus b)$ [by absorption law]

$= a * b$

$\Rightarrow a * b = a$

$\therefore a * b = a \Leftrightarrow a \oplus b = b$

Hence $a \leq b \Leftrightarrow a * b = a \Leftrightarrow a \oplus b = b$.

Theorem:

Let $(L, \leq)$ be a lattice. For any $a, b, c \in L$, the following inequalities called the distributive inequalities hold:

(i) $a \oplus (b * c) \leq (a \oplus b) * (a \oplus c)$

(ii) $a * (b \oplus c) \geq (a * b) \oplus (a * c)$

Proof of (i):

Let $a, b, c \in L$. As $a \leq (a \oplus b)$ and $a \leq (a \oplus c)$

we have $a \leq (a \oplus b) * (a \oplus c)$ — (1)

[$\because a \leq b$ & $a \leq c \Rightarrow a \leq b * c$]

As $b * c \leq b \leq a \oplus b$ &

$b * c \leq c \leq a \oplus c$

we have

$b * c \leq (a \oplus b) * (a \oplus c)$ — (2)

[$\because a \leq b$ & $a \leq c \Rightarrow a \leq b * c$]

from (1) & (2)

$\boxed{a \oplus b * c \leq (a \oplus b) * (a \oplus c)}$ [$\because a \geq b$ & $a \geq c \Rightarrow a \geq b \oplus c$]

Proof of (ii):

$a \geq a * b$ & $a \geq a * c$

we have $a \geq (a * b) \oplus (a * c)$ — (3)

As $b \oplus c \geq b \geq a * b$

$b \oplus c \geq c \geq a * c$

We have $b \oplus c \geq (a*b) \oplus (a*c)$ —— (4)

From (3) & (4) $\quad [\because a \geq b \ \& \ a \geq c \Rightarrow a \geq b \oplus c]$

$$\boxed{a*(b \oplus c) \geq (a*b) \oplus (a*c)}$$

$[\because a \leq b \ \& \ a \leq c \Rightarrow \cancel{a \leq b*} \ a \leq (b*c)]$

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