TOPIC:10-Boolean algebra

Boolean Algebra:

A Boolean algebra is a complemented, distributive lattice.

Note: A Boolean algebra will generally be denoted by $(B, *, \oplus, ', 0, 1)$ in which $(B, *, \oplus)$ is a lattice with two Binary operation $*$ & $\oplus$. where '0' is the least element & '1' is the greatest element and ' ' ' denotes the complementation.

Note: $(B, *, \oplus, 0, 1)$ is a bounded lattice in which for any $a \in B$, the following hold.

(i) $0 \leq a \leq 1$

(2) $a * 0 = 0$, $a \oplus 1 = 1$

(3) $a * 1 = a$, $a \oplus 0 = a$

Note: $(B, *, \oplus, ', 0, 1)$ is a uniquely complemented lattice in which complement of any element $a \in B$ is denoted by $a' \in B$ & satisfies the following identities.

(1) $a * a' = 0$; $a \oplus a' = 1$

(2) $0' = 1$; $1' = 0$

(3) $(a * b)' = a' \oplus b'$; $(a \oplus b)' = a' * b'$

Sub-Boolean algebra:

Let $(B, *, \oplus, ', 0, 1)$ be a Boolean algebra and $S \subseteq B$. If S contains the elements 0 & 1 and is closed under the operations $*$, $\oplus$ & $'$, then $(S, *, \oplus, ', 0, 1)$ is called a Sub-Boolean algebra.

Show that in any Boolean algebra
$$(a+b)\cdot(a'+c) = ac + a'b + bc.$$

Soln:

Let $(B, \oplus, *, ', 0, 1)$ be a Boolean algebra & Let $a, b, c \in B$

$$LHS = (a+b)\cdot(a'+c) = (a+b)a' + (a+b)\cdot c$$
$$= aa' + ba' + ac + bc$$
$$= 0 + a'b + ac + bc$$
$$= ac + a'b + bc = RHS.$$

In a Boolean algebra $(L, \oplus, *)$ (i.e.) complemented & distributive lattice. Show that the complement of any element $a \in L$ is unique.

Proof:

Let $a \in L$ has two complements b & $c \in L$

By def. $a * b = 0$, $a \oplus b = 1$

$a * c = 0$, $a \oplus c = 1$

We have $b = b * 1 = b * (a \oplus c)$

$= (b * a) \oplus (b * c)$

$= 0 \oplus (b * c)$

$b = b * c$ — (1)

$b = b * c$ — (1)

$c = c * 1 = c * (a \oplus b)$

$= (c * a) \oplus (c * b)$

$= 0 \oplus (c * b)$

$= (c * b)$

$= b * c$ — (2)

By (1) & (2) $b = c$

∴ Every element of L has a unique complement.

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