Stat and prove Demorgan's laws in Boolean algebra.

Stat:

Let $(L, \oplus, *)$ be Boolean lattice.

(ie) L is complemented and distributive lattice.

The Demorgan's law are

(i) $\overline{a \oplus b} = \bar{a} * \bar{b}$ (ii) $\overline{a * b} = \bar{a} \oplus \bar{b}$, $\forall a, \bar{a}, b, \bar{b} \in L$.

Proof:

Assume that $a, b \in L$. Such that $a \oplus \bar{a} = 1$; $a * \bar{a} = 0$

$b \oplus \bar{b} = 1$; $b * \bar{b} = 0$.

(i) To prove: $\overline{a \oplus b} = \bar{a} * \bar{b}$

We have to show that $(a \oplus b) \oplus (\bar{a} * \bar{b}) = 1$

& $(a \oplus b) * (\bar{a} * \bar{b}) = 0$.

Now
$$\begin{aligned}
(a \oplus b) \oplus (\bar{a} * \bar{b}) &= [(a \oplus b) \oplus \bar{a}] * [(a \oplus b) \oplus \bar{b}] \\
&= [(a \oplus \bar{a}) + b] * [(a \oplus b) \oplus \bar{b}] \\
&= [1 \oplus b] * [a \oplus 1] \\
&= 1 * 1 = 1
\end{aligned}$$

$$\begin{aligned}
(a \oplus b) * (\bar{a} * \bar{b}) &= [(a \oplus b) * \bar{a}] * [(a \oplus b) * \bar{b}] \\
&= [(a * \bar{a}) \oplus (b * \bar{a})] * [(a * \bar{b}) \oplus (b * \bar{b})] \\
&= [0 \oplus (b * \bar{a})] * [(a * \bar{b}) \oplus 0] \\
&= (b * \bar{a}) * (\bar{a} * \bar{b}) \\
&= b * (\bar{a} * a) * b
\end{aligned}$$

$$= b * (\bar{a} * a) * b$$
$$= b * 0 * b$$
$$= 0$$

ii) To prove $\overline{a * b} = \bar{a} \oplus \bar{b}$

We have to show that $(a*b) \oplus (\bar{a} \oplus \bar{b}) = 1$

& $(a*b) * (\bar{a} \oplus \bar{b}) = 0$

Now $(a*b) \oplus (\bar{a} \oplus \bar{b}) = [(a*b) \oplus \bar{a}] \oplus [(a*b) \oplus \bar{b}]$

$$= [(a \oplus \bar{a}) * (b \oplus \bar{a})] \oplus [(a \oplus \bar{b}) * (b \oplus \bar{b})]$$
$$= [1 * (b \oplus \bar{a})] \oplus [(a \oplus \bar{b}) * 1]$$
$$= (b \oplus \bar{a}) \oplus (a \oplus \bar{b})$$
$$= b \oplus (\bar{a} \oplus a) \oplus \bar{b}$$
$$= b \oplus 1 + \bar{b}$$
$$= 1$$

$(a*b) * (\bar{a} \oplus \bar{b}) = [(a*b) * \bar{a}] \oplus [(a*b) * \bar{b}]$

$$= [(a * \bar{a}) * b] \oplus [a * (b * \bar{b})]$$
$$= [0 * b] \oplus [a * 0]$$
$$= 0 \oplus 0$$
$$= 0$$

Hence the Demorgan's laws are proved

.

Show that in a complemented and distributive lattice $a \leq b \Leftrightarrow a * b' = 0 \Leftrightarrow a' \oplus b = 1 \Leftrightarrow b' \leq a'$

Solution:

(i) $\Rightarrow$ (ii)

$a \leq b \Rightarrow a \oplus b = b$

$\Rightarrow (a \oplus b) * b' = 0$ [as $b * b' = 0$]

$\Rightarrow (a * b') \oplus (b * b') = 0$

$\Rightarrow (a * b') \oplus 0 = 0$ [as $b * b' = 0$]

$\Rightarrow a * b' = 0$

Hence $a \leq b \Rightarrow a * b' = 0$.

(ii) $\Rightarrow$ (iii)

$a * b' = 0$

$\Rightarrow (a * b')' = 1$

$\Rightarrow a' \oplus (b')' = 1$

$\Rightarrow a' \oplus b = 1$

Hence $a * b' = 0 \Rightarrow a' \oplus b = 1$

(iii) $\Rightarrow$ (iv)

$a' \oplus b = 1 \Rightarrow (a' \oplus b) * b' = b'$

$\Rightarrow (a' * b') \oplus (b * b') = b'$ [distributive law]

$\Rightarrow a' * b' = b'$ [as $b * b' = 0$]

$\Rightarrow b' \leq a'$

Hence $a' \oplus b = 1 \Rightarrow b' \leq a'$

(iv) $\Rightarrow$ (iv)

$b' \leq a' \Rightarrow a' * b' = b'$

$\Rightarrow (a' * b')' = (b')'$

$= a \oplus b = b$

$\Rightarrow a \leq b$

Hence $b' \leq a' \Rightarrow a \leq b$

Thus (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) $\Rightarrow$ (i)

In a Boolean algebra, prove that $a.(a \oplus b) = a$. for all $a, b \in B$.

Solu:

$$a.(a+b) = (a+0).(a+b) \quad \text{[by identity law]}$$
$$= a+(0.b) \quad \text{[by distributive]}$$
$$= a+0 \quad [\because 0.b = 0]$$
$$= a \; //$$

Simplify the Boolean expression. $a'.b'.c + a.b'.c + a'.b'.c'$ using Boolean algebra identities.

Solu:

Given $a'.b'.c + a.b'.c + a.b'.c'$

$$= a'.b'.c + a.b'.(c+c') \quad \text{[by distributive]}$$
$$= \underline{a'.b'}.c + \underline{a.b'}.1$$
$$= \cancel{b'.a'.c + b'.a.1} \quad \text{[} \cancel{\text{by commutative}} \text{]}$$
$$= \cancel{b'.((c'+a)}$$

$= b'.a'.c + b'.a.1$ [by commutative]

$= b'.(a'.c + a.1)$

$= b'.(a + a'.c)$

$= b'.(a+a').(a+c)$

$= b'.1.(a+c)$

$= b'.(a+c)$

$= a.b' + b'.c$