TOPIC:7- Lattice Homomorphism

Lattice homomorphism:

Let $(L, *, \oplus)$ and $(S, \wedge, \vee)$ be two lattices. A mapping $g : L \rightarrow S$ is called lattice homomorphism from the lattice $(L, *, \oplus)$ to $(S, \wedge, \vee)$ if for $a, b \in L$

$$g(a * b) = g(a) \wedge g(b) \quad \&$$
$$g(a \oplus b) = g(a) \vee g(b)$$

Every Chain is a distributive lattice.

Proof:

Let $(L, \leq)$ be a chain

Let $a, b, c \in L$

To prove: $a * (b \oplus c) = (a * b) \oplus (a * c)$

Case (i) Suppose that $a \leq b$ & $a \leq c$.

then we have $a * b = a$, $a \oplus a = a$ & $a * c = a$

$\Rightarrow \quad a \leq b \oplus c \quad [\because a \leq b \text{ \& } a \leq c \Rightarrow a \leq b \oplus c]$

So $a * (b \oplus c) = a$ —— (1)

& $(a * b) \oplus (a * c) = a \oplus a = a$ —— (2)

From (1) & (2) we get

$$\boxed{a * (b \oplus c) = (a * b) \oplus (a * c)}$$

Case (ii)

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Suppose that $a \geq b$ & $a \geq c$ then we have

$a * b = b$ $\quad a * c = c$ & ~~$b \oplus c \leq a$~~

& $a \geq b \oplus c$ $\quad$ [$\because a \geq b$ & $a \geq c \Rightarrow a \geq b \oplus c$]

So that $a * (b \oplus c) = b \oplus c$ —— (3)

$(a * b) \oplus (a * c) = b \oplus c$ —— (4)

From (3) & (4) we get

$$\boxed{a * (b \oplus c) = (a * b) \oplus (a * c)}$$

Using principle of duality in both cases.

The other form of distributive law

$a \oplus (b * c) = (a \oplus b) * (a \oplus c)$ also hold.

$\therefore$ Every chain is a distributive lattice.

In a distributive lattice, show that

$(a*b)\oplus(b*c)\oplus(c*a) = (a\oplus b)*(b\oplus c)*(c\oplus a)$

Proof:

LHS $= (a*b)\oplus(b*c)\oplus(c*a)$

$= (a*b)\oplus[((b*c)\oplus c)*((b*c)\oplus a)]$ [by distributive]

$= (a*b)\oplus[c*(a\oplus(b*c))]$ [by absorption law]

$= [(a*b)\oplus c]*[(a*b)\oplus a\oplus(a*b)\oplus(b*c)]$

$= [(a\oplus c)*(b\oplus c)]*[a\oplus(a*b)\oplus(b*c)]$

$= (a\oplus c)*(b\oplus c)*[a\oplus(b*c)]$

$= (a\oplus c)*(b\oplus c)*(a\oplus b)*(a\oplus c)]$

$= (a\oplus b)*(b\oplus c)*(c+a)$

$=$ RHS.

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