Topic: 3. 4 – CIRCLE OF CURVATURE

Equation to the circle of curvature.

Let $(\bar{x}, \bar{y})$ be the centre of curvature and ρ be the radius of curvature corresponding to a point (x, y) of the given curve. The equation of the circle of curvature is

$$(x-\bar{x})^2 + (y-\bar{y})^2 = \rho^2.$$

1. find the circle of curvature at the point $(a/4, a/4)$ of the curve $\sqrt{x} + \sqrt{y} = \sqrt{a}$

soln:

circle of curvature formula is

$$(x-\bar{x})^2 + (y-\bar{y})^2 = \rho^2$$

where $\bar{x} = x - \frac{y_1}{y_2}(1+y_1^2)$

$$\bar{y} = y + \frac{(1+y_1^2)}{y_2}.$$

Given $\sqrt{x} + \sqrt{y} = \sqrt{a}$.

Diff. w.r.t x.

$$\frac{1}{2\sqrt{x}} + \frac{1}{2\sqrt{y}} \cdot \frac{dy}{dx} = 0.$$

$$\Rightarrow y_1 = -\sqrt{\frac{y}{x}} = \frac{-\sqrt{y}}{\sqrt{x}}.$$

$$y_{1(a/4,\, a/4)} = -\sqrt{\frac{a/4}{a/4}} = -1.$$

$$y_2 = -\left[\frac{\sqrt{x} \cdot \frac{1}{2\sqrt{y}}\left(\frac{dy}{dx}\right) - \sqrt{y}\left(\frac{1}{2\sqrt{x}}\right)}{x}\right]$$

$$y_2(a/4, a/4) = -\left[\frac{\sqrt{a/4} \cdot \frac{1}{2\sqrt{a/4}}(-1) - \sqrt{\frac{a}{4}} \cdot \frac{1}{2\sqrt{\frac{a}{4}}}}{a/4}\right]$$

$$= \frac{-\left[-\frac{1}{2} - \frac{1}{2}\right]}{a/4} = 4/a.$$

$$\rho = \frac{\left(1 + y_1^2\right)^{3/2}}{y_2} = \frac{(1+1)^{3/2}}{4/a} = 2^{3/2} \cdot \frac{a}{4}$$

$$\rho = \frac{a}{\sqrt{2}}$$

$$\bar{x} = x - \frac{y_1}{y_2}\left(1 + y_1^2\right)$$

$$\bar{x}_{(a/4,\, a/4)} = \frac{a}{4} - \frac{(-1)}{4/a}\left(1+(-1)^2\right)$$

$$= \frac{a}{4} + \frac{a}{4}(2) = \frac{2a+a}{4} = \frac{3a}{4}$$

$$\bar{y} = y + \frac{(1+y_1^2)}{y_2}$$

$$\bar{y}_{(a/4,\, a/4)} = \frac{a}{4} + \frac{(1+1)}{4/a} = \frac{a}{4} + \frac{a}{4}(2) = \frac{3a}{4}$$

The centre of curvature is $\left(\frac{3a}{4}, \frac{3a}{4}\right)$

∴ The equation of centre of curvature is

$$(x-\bar{x})^2 + (y-\bar{y})^2 = \rho^2$$

$$\left(x - \frac{3a}{4}\right)^2 + \left(y - \frac{3a}{4}\right)^2 = \frac{a^2}{2}.$$

2. Find the equation of the circle of curvature at (c, c) on $xy = c^2$.

Soln: circle of curvature formula is

$$(x-\bar{x})^2 + (y-\bar{y})^2 = \rho^2$$

Given $xy = c^2$.

$$y = \frac{c^2}{x} \Rightarrow y_1 = \frac{dy}{dx} = \frac{-c^2}{x^2}.$$

$$y_2 = \frac{2c^2}{x^3}.$$

$$y_{1(c,c)} = -\frac{c^2}{c^2} = -1 \;;\; y_{2}(c,c) = \frac{2c^2}{c^3} = \frac{2}{c}$$

$$\rho = \frac{(1+y_1^2)^{3/2}}{y_2} = \left[1+(-1)^2\right]^{3/2} \cdot \frac{c}{2} = 2^{3/2} \cdot \frac{c}{2}$$

$$\rho = c\sqrt{2}$$

$$\bar{x} = x - \frac{y_1}{y_2}(1+y_1^2)$$

$$\bar{x}_{(c,c)} = c - (-1)\frac{c}{2}\left[1+(-1)^2\right] = c + \frac{2c}{2} = 2c.$$

$$\bar{y} = y + \frac{(1+y_1^2)}{y_2} = c + \frac{(1+(-1)^2)}{2} \cdot c$$

$$= c + \frac{2c}{2} = 2c.$$

$\therefore$ the equation of circle is $(x-2c)^2 + (y-2c)^2 = 2c^2$.