



AN AUTONOMOUS INSTITUTION

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Topic: 3.9 – ENVELOPES

Problems - Envelope of two parameter family of curves.

1. Find the envelope of the family of lines $\frac{x}{a} + \frac{y}{b} = 1$
where the parameters a and b are connected by
the relation $a + b = c$.

Soln: Given $\frac{x}{a} + \frac{y}{b} = 1 \rightarrow (1)$

Taking differential we get.

$$-\frac{x}{a^2} da - \frac{y}{b^2} db = 0 \rightarrow (2)$$

Given $a + b = c \rightarrow (3)$

Taking differential we get

$$da + db = 0 \rightarrow (4)$$

$$(2) \Rightarrow \frac{x}{a^2} da = -\frac{y}{b^2} db$$

$$(4) \Rightarrow da = -db$$

$$\frac{(2)}{(4)} \Rightarrow \frac{\frac{x}{a^2} da}{da} = \frac{-\frac{y}{b^2} db}{-db}$$

$$\frac{x^2}{a^2} = \frac{y^2}{b^2}$$

$$\frac{x/a}{a} = \frac{y/b}{b} = \frac{x/a + y/b}{a+b} = \frac{1}{c}$$

$$(i.e) \frac{x}{a^2} = \frac{1}{c} ; \frac{y}{b^2} = \frac{1}{c}$$

$$\Rightarrow a^2 = xc ; b^2 = yc.$$



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$$a = (xc)^{1/2} \quad ; \quad b = (yc)^{1/2}$$

③ $\Rightarrow a + b = c$
 $(xc)^{1/2} + (yc)^{1/2} = c$
 $\sqrt{x}\sqrt{c} + \sqrt{y}\sqrt{c} = c$
 $\sqrt{c}[\sqrt{x} + \sqrt{y}] = c$
 $\sqrt{x} + \sqrt{y} = \frac{c}{\sqrt{c}}$
 $\sqrt{x} + \sqrt{y} = \sqrt{c}$

Q. Find the envelope of the family of lines $\frac{x}{a} + \frac{y}{b} = 1$
where the parameters a and b are connected
by the relation $a^2 + b^2 = c^2$

Soln: Given $\frac{x}{a} + \frac{y}{b} = 1 \rightarrow$ ①

Taking differentials; $-\frac{x}{a^2} da - \frac{y}{b^2} db = 0 \rightarrow$ ②

Given $a^2 + b^2 = c^2 \rightarrow$ ③

Taking differential $2ada + 2bdb = 0 \rightarrow$ ④

② $\Rightarrow \frac{x}{a^2} da = -\frac{y}{b^2} db \rightarrow$ ⑤

④ $\Rightarrow 2ada = -2bdb \rightarrow$ ⑥

⑤ / ⑥ $\Rightarrow \frac{x/a^2}{a} = \frac{y/b^2}{b} \Rightarrow \frac{x/a + y/b}{a^2 + b^2} = \frac{1}{c^2}$



$$\begin{aligned} \frac{x}{a^3} &= \frac{1}{c^2} \quad ; \quad \frac{y}{b^3} = \frac{1}{c^2} \\ \Rightarrow a^3 &= xc^2 \quad ; \quad b^3 = yc^2 \\ a &= (xc^2)^{1/3} \quad \quad b = (yc^2)^{1/3} \\ \text{Given } a^2 + b^2 &= c^2 \\ (xc^2)^{2/3} + (yc^2)^{2/3} &= c^2 \\ (c^2)^{2/3} [x^{2/3} + y^{2/3}] &= c^2 \\ x^{2/3} + y^{2/3} &= c^{2-4/3} \\ x^{2/3} + y^{2/3} &= c^{2/3} \end{aligned}$$

3. Find the envelope of $\frac{x}{l} + \frac{y}{m} = 1$ where l and m are connected by $\frac{l}{a} + \frac{m}{b} = 1$ and a, b are constants.

Soln: Given $\frac{x}{l} + \frac{y}{m} = 1 \rightarrow \textcircled{1}$

Taking differentials.

$$-\frac{x}{l^2} dl - \frac{y}{m^2} dm = 0 \rightarrow \textcircled{2}$$

Given $\frac{l}{a} + \frac{m}{b} = 1 \rightarrow \textcircled{3}$

Taking differential $\frac{dl}{a} + \frac{dm}{b} = 0 \rightarrow \textcircled{4}$



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(2) $\Rightarrow \frac{x}{l^2} dl = -\frac{y}{m^2} dm \rightarrow (5)$

(4) $\Rightarrow \frac{dl}{a} = -\frac{dm}{b} \rightarrow (6)$

(5) $\Rightarrow \frac{ax}{l^2} = -\frac{by}{m^2}$

(6) $\Rightarrow \frac{x/l}{y/a} = \frac{-y/m}{m/b} = \frac{x/l + y/m}{l/a + m/b} = \frac{1}{1} = 1$

$\frac{xa}{l^2} = 1 ; \frac{yb}{m^2} = 1$
 $xa = l^2 \quad yb = m^2$

(i-e) $l = \sqrt{ax} ; y = \sqrt{by}$

Given $\frac{l}{a} + \frac{m}{b} = 1$

$\frac{\sqrt{ax}}{a} + \frac{\sqrt{by}}{b} = 1$

$\sqrt{\frac{x}{a}} + \sqrt{\frac{y}{b}} = 1$ which is the required envelope.