CONGRUENCES : (2 m)

Def : Let m be a positive integer. An integer a is congruent to an integer b modulo m if $m | (a-b)$. Symbolically, we write $a \equiv b \pmod{m}$. Here, m is the modulus of the congruence relation.

Example :

(i) congruence mod 12 to tell the time of the day

(ii) congruence mod 7 to tell the day of the week

2m

Theorem :

$a \equiv b \pmod{m}$ if and only if $a = b + mk$ for some integer k.

Proof :

Let $a \equiv b \pmod{m}$

Then $m | (a-b)$

$\Rightarrow a - b = mk$ for some integer k.

$\Rightarrow a = b + mk$

Conversely, let $a = b + mk$

Then $a - b = mk$

$\Rightarrow m | (a-b)$

$\Rightarrow a \equiv b \pmod{m}$.

Example 1:

Find the remainder $1! + 2! + 3! + \dots + 100!$ is divided by 15.

Solution:

For divisibility by 15 we consider mod 15

$5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$ is divisible by 15.

Therefore $5! \equiv 0 \pmod{15}$.

$\Rightarrow r! \equiv 0 \pmod{15}$ for $r \geq 5$.

Now, $1! + 2! + 3! + \dots 100! = 1! + 2! + 3! + 4! + 0 + 0 + \dots + 0 \pmod{15}$.

$$\equiv 1 + 2 + 6 + 24 \pmod{15}$$

$$= 33 \pmod{15}$$

$$= 3 \pmod{15}$$

Therefore when $1! + 2! + 3! + \dots 100!$ is divided by 15, the remainder is 3.

2. Find the remainder when 3^{247} is divided by 17.

Solution :

we have $3^3 \equiv 27 \equiv 10 \pmod{17}$

$3^6 \equiv 10^2 \pmod{17}$

$\equiv 15 \pmod{17}$

$\equiv -2 \pmod{17}$

$(3^6)^4 \equiv (-2)^4 \pmod{17}$

$3^{24} \equiv 16 \pmod{17}$

$\equiv -1 \pmod{17}$

$(3^{24})^{10} \equiv (-1)^{10} \pmod{17}$

$\equiv 1 \pmod{17}$

Now, $3^{247} = 3^{240+6+1}$

$= 3^{240} \cdot 3^{6} \cdot 3^{1}$

$3^{247} = 1 \cdot (-2) \cdot 3 \pmod{7}$.

$= -6 \pmod{17}$

$3^{247} \equiv 11 \pmod{17}$

Hence, when 3^{247} is divided by 17 the remainder is 11.

Find the remainder when 13^{218} is divided by 17.

solution :

we have $13^{2} = 169 \equiv 16 \pmod{17}$

$13^{2} \equiv -1 \pmod{17}$

$(13^{2})^{109} \equiv (-1)^{109} \pmod{17}$

$13^{218} \equiv -1 \pmod{17}$

$13^{218} \equiv 16 \pmod{17}$

Hence, when 13^{218} is divided by 17 the remainder is 16.