



TOPIC 9: STEADY STATE SOLUTIONS OF TWO DIMENSIONAL HEAT EQUATION

Write down all possible sol. of two dimensional heat eqn.

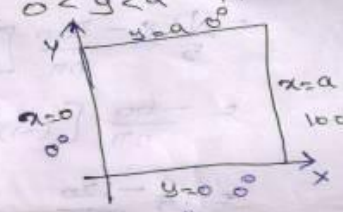
① $u(x,y) = (A \cos px + B \sin px) (C e^{py} + D e^{-py})$

② $u(x,y) = (A e^{px} + B e^{-px}) (C \cos py + D \sin py)$

③ $u(x,y) = (Ax + B)(Cy + D)$

A Square plate is bounded by the lines $x=0$ $y=0$ $x=a$ $y=a$. The lines $x=0$ $y=0$ $x=a$ are kept at 0°C . The side $x=a$ is kept at temp. 100°C . Find $u(x,y)$.

The temp $u(x,y)$ is from

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$


The Cond. are

① $u(x,0) = 0$

② $u(x,a) = 0$

③ $u(0,y) = 0$

④ $u(a,y) = 100$

The suitable sol. is

$$u(x,y) = (A e^{px} + B e^{-px}) (C \cos py + D \sin py) \quad \text{--- (I)}$$

Apply ①

$$0 = (A e^{px} + B e^{-px}) (C)$$
$$\Rightarrow \boxed{C=0}$$

Apply ② Sub $C=0$ in I

$$u(x,y) = (A e^{px} + B e^{-px}) (D \sin py) \quad \text{--- (II)}$$

Apply ③ in II



$D \sin pa = 0$
 $\sin pa = 0 = \sin n\pi$
 $P = \frac{n\pi}{a}$

Sub $P = \frac{n\pi}{a}$ in II

$u(m, y) = (A e^{\frac{n\pi x}{a}} + B e^{-\frac{n\pi x}{a}}) (D \sin \frac{n\pi y}{a}) - \frac{100}{a}$

APPLY (iii)

$u(0, y) = (A + B) (D \sin \frac{n\pi y}{a}) = 0$
 $B = -A$

\therefore (2) Bernoulli

$u(m, y) = A (e^{\frac{n\pi x}{a}} - e^{-\frac{n\pi x}{a}}) D \sin \left(\frac{n\pi y}{a} \right)$

The most gen. sol. is

$u(m, y) = \sum_{n=1}^{\infty} D_n (e^{\frac{n\pi x}{a}} - e^{-\frac{n\pi x}{a}}) \sin \left(\frac{n\pi y}{a} \right) - \frac{100}{a}$

APPLY (iv) in I

$u(a, y) = \sum_{n=1}^{\infty} D_n (e^{n\pi} - e^{-n\pi}) \sin \left(\frac{n\pi y}{a} \right) = 100$

$D_n (e^{n\pi} - e^{-n\pi}) = \frac{200}{a} \int_0^a 100 \sin \left(\frac{n\pi y}{a} \right) dy$

$= \frac{200}{a} \int_0^a \sin \left(\frac{n\pi y}{a} \right) dy$

$= \frac{200}{a} \left[-\cos \left(\frac{n\pi y}{a} \right) \left(\frac{a}{n\pi} \right) \right]_0^a$

$= \frac{200}{a} \left[+\frac{a}{n\pi} (-1)^n + \frac{a}{n\pi} \right]$

$= \frac{200}{a} \times \frac{a}{n\pi} [-(-1)^n + 1]$

$D_n [e^{n\pi} - e^{-n\pi}] = \begin{cases} 0 & n = 2, 4, 6, \dots \\ \frac{400}{n\pi} & n = 1, 3, 5, \dots \end{cases}$



Handwritten mathematical derivation on a whiteboard:

$$D_n = \frac{400}{n\pi [e^{n\pi} - e^{-n\pi}]} \quad n=1,3,5,\dots$$
$$u(x,y) = \sum_{n=1,3,5,\dots}^{\infty} \frac{400}{n\pi [e^{n\pi} - e^{-n\pi}]} \left(e^{\frac{n\pi y}{a}} - e^{-\frac{n\pi y}{a}} \right) \sin\left(\frac{n\pi x}{a}\right)$$