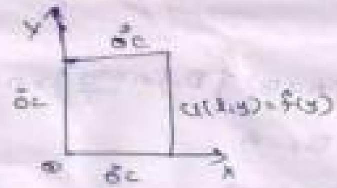
TOPIC 10: STEADY STATE SOLUTIONS OF TWO DIMENSIONAL HEAT EQUATION

Finite Plate with Value given in y-direction.

(1) $u(x,0)=0$
(2) $u(x,l)=0$
(3) $u(0,y)=0$
(4) $u(l,y)=f(y)$

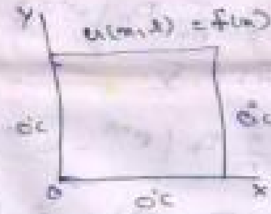


The Suitable Sol.

$u(x,y)=(Ae^{px}+Be^{-px})(C\cos py+D\sin py)$

x - Direction

(1) $u(0,y)=0$
(2) $u(l,y)=0$
(3) $u(x,0)=0$
(4) $u(x,l)=f(x)$



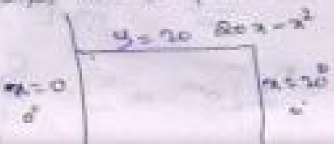
The Suitable Sol.

$u(x,y)=(A\cos px+B\sin px)(Ce^{py}+De^{-py})$

—x—

A Sqr plate is bounded by the lines $x=0, y=0$, $x=20$ and $y=20$. Its faces are insulated. The tempr. along the upper horizontal edge is gn. by $u(x,20)=x(20-x)=20x-x^2$, $0<x<20$ while the other edges are kept at 0°C. Find the steady state tempr. distr. in the plate.

Sol:



The temp. $u(x,y)$ is from

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

The condn. are

(1) $u(0,y)=0$ (2) $u(20,y)=0$ (3) $u(x,0)=0$ (4) $u(x,20) = 20x - x^2$

The suitable soln. is

$$u(x,y) = [A\cos px + B\sin px][Ce^{py} + De^{-py}]$$

Apply (1) $0 = A(Ce^{py} + De^{-py}) \Rightarrow \boxed{A=0}$

Apply (2) $0 = B\sin(20p)(Ce^{py} + De^{-py})$

$B\sin(20p) = 0 \quad 20p = n\pi \Rightarrow p = n\pi/20$

Apply (3)

$0 = B\sin px (C+D)$

$C + D = 0 \Rightarrow D = -C$

$\therefore$ (1) Becomes

$u(x,y) = B\sin px\, C[e^{py} - e^{-py}]$

$= B\sin\left(\frac{n\pi x}{20}\right) C\left[e^{\frac{n\pi y}{20}} - e^{-\frac{n\pi y}{20}}\right]$

Most genl. soln.

$$u(x,y) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi x}{20}\right)\left[e^{\frac{n\pi y}{20}} - e^{-\frac{n\pi y}{20}}\right]$$

Apply (4)

$$20x - x^2 = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi x}{20}\right)\left[e^{n\pi} - e^{-n\pi}\right]$$

$$C_n\left[e^{n\pi} - e^{-n\pi}\right] = \frac{2}{20}\int_0^{20}(20x - x^2)\sin\left(\frac{n\pi x}{20}\right)dx$$

$$= \frac{1}{10}\left[\frac{-20(20x - x^2)}{n\pi}\cos\left(\frac{n\pi x}{20}\right) + \frac{400}{n^2\pi^2}(20 - 2x)\sin\left(\frac{n\pi x}{20}\right) - \frac{16000}{n^3\pi^3}\cos\left(\frac{n\pi x}{20}\right)\right]_0^{20}$$

$$= \frac{1}{10}\left[\frac{-16000}{n^3\pi^3}(-1)^n + \frac{16000}{n^3\pi^3}(1)\right]$$

$$= \frac{1600}{n^3\pi^3}\left[-(-1)^n + 1\right]$$

$$= \begin{cases} 0 & n = 2, 4, 6 \ldots \\ \frac{3200}{n^3\pi^3} & n = 1, 3, 5 \ldots \end{cases}$$

$$C_n = \frac{3200}{n^3\pi^3(e^{n\pi} - e^{-n\pi})}$$

$$u(x,y) = \sum_{n=\text{odd}}^{\infty} \frac{3200}{n^3\pi^3(e^{n\pi} - e^{-n\pi})}\sin\left(\frac{n\pi x}{20}\right)\left(e^{\frac{n\pi y}{20}} - e^{-\frac{n\pi y}{20}}\right)$$