



TOPIC 9: Initial Value and Final value Theorem

Initial value Theorem

If  $\mathcal{Z}[f(t)] = F(z)$  then

$$f(0) = \lim_{z \rightarrow \infty} F(z)$$

Proof:

Given,  $F(z) = \mathcal{Z}[f(t)] = \sum_{n=0}^{\infty} f(nT) z^{-n}$

$$F(z) = f(0)z^{-0} + f(1T)z^{-1} + f(2T)z^{-2} + \dots$$

$$= f(0) + \frac{f(1T)}{z} + \frac{f(2T)}{z^2} + \dots$$

$$\lim_{z \rightarrow \infty} F(z) = \lim_{z \rightarrow \infty} \left[ f(0) + \frac{f(1T)}{z} + \frac{f(2T)}{z^2} + \dots \right]$$

$$\lim_{z \rightarrow \infty} F(z) = f(0)$$


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Final value Theorem

If  $\mathcal{Z}[f(t)] = F(z)$ ,

$$\lim_{t \rightarrow \infty} f(t) = \lim_{z \rightarrow 1} (z-1) F(z)$$

Proof:



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$$\begin{aligned}
 z [f(t+T) - f(t)] &= \sum_{n=0}^{\infty} [f(nT+T) - f(nT)] z^{-n} \\
 z [f(t+T)] - z [f(t)] &= \sum_{n=0}^{\infty} [f(nT+T) - f(nT)] z^{-n} \\
 z F[z] - z f(0) - F(z) &= \sum_{n=0}^{\infty} [f(nT+T) - f(nT)] z^{-n} \\
 (z-1) F[z] - z f(0) &= \sum_{n=0}^{\infty} [f(nT+T) - f(nT)] z^{-n} \\
 \lim_{z \rightarrow 1} (z-1) F(z) - z f(0) &= \lim_{z \rightarrow 1} \sum_{n=0}^{\infty} [f(nT+T) - f(nT)] z^{-n} \\
 &= \sum_{n=0}^{\infty} [f(nT+T) - f(nT)] \\
 &= \lim_{n \rightarrow \infty} [f(T) - f(0) + f(2T) - f(T) + \dots + f(nT) - f((n-1)T)] \\
 &= \lim_{n \rightarrow \infty} f(nT) - f(0) \\
 \lim_{z \rightarrow 1} (z-1) F(z) - \lim_{z \rightarrow 1} z f(0) &= \lim_{z \rightarrow 1} \sum_{n=0}^{\infty} [f(nT+T) - f(nT)] (z-1)^{-n} \\
 \lim_{z \rightarrow 1} (z-1) F(z) - f(0) &= \sum_{n=0}^{\infty} [f(nT+T) - f(nT)] \\
 &= \lim_{n \rightarrow \infty} [f(nT) - f(0) + f((n+1)T) - f(nT)] \\
 &= \lim_{n \rightarrow \infty} f((n+1)T) - f(0) \\
 \lim_{z \rightarrow 1} (z-1) F(z) - f(0) &= f(\infty) - f(0)
 \end{aligned}$$



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