TOPIC 11: Solution of difference equation

Solution of difference equation using Z-transform

To slove difference equ we need following results,

$$Z\left[y(n+1)\right] = z\,F[z] - z\,y(0)$$

$$Z\left[y(n+2)\right] = z^2 F[z] - z^2 y(0) - z\,y(1)$$

$$Z\left[y(n+3)\right] = z^3 F[z] - z^3 y(0) - z^2 y(1) - z\,y(2)$$

where, $Z\left[(y^n)\right] = F[z]$

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1, Solve : $y_{n+2} + 6y_{n+1} + 9y_n = 2^n$

Given :

$y_0 = y_1 = 0$

Solution :

$y_{n+2} + 6y_{n+1} + 9y_n = 2^n$

$Z[y_{n+2}] + 6Z[y_{n+1}] + 9Z[y_n] = Z[2^n]$

$[z^2 F(z) - z^2 y(0) - z y(1)] + 6[zF[z] - z y[0]] + 9F[z] = \frac{z}{z-2}$

$[z^2 F(z) - 0 - 0] + 6[zF(z) - 0] + 9F(z) = \frac{z}{z-2}$

$z^2 F(z) + 6zF(z) + 9F[z] = \frac{z}{z-2}$

$[z^2 + 6z + 9]F[z] = \frac{z}{z-2}$

$(z+3)^2 F(z) = \frac{z}{z-2}$

$F(z) = \frac{z}{(z-2)(z+3)^2}$

$\frac{F(z)}{z} = \frac{1}{(z-2)(z+3)^2} = \frac{A}{z-2} + \frac{B}{z+3} +$

$\frac{1}{(z-2)(z+3)^2} = \frac{A(z+3)^2 + B(z+3)(z-2) + C(z}{(z-2)(z+3)^2}$

$1 = A(z+3)^2 + B(z+3)(z-2) + C(z-2)$

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put $z = -3$

$1 = 0 + 0 + C(-5)$

$1 = -5C$

$\boxed{C = -\frac{1}{5}}$

put $z = 2$

$1 = A(5)^2 + 0 + 0$

$1 = 25A$

$\boxed{A = \frac{1}{25}}$

Put $z = 0$

$1 = 9A + (-6)B + (-2)C$

$1 = 9 \times \frac{1}{25} - 6B - 2(-\frac{1}{5})$

$1 = \frac{9}{25} - 6B + \frac{2}{5}$

$1 = \frac{9}{25} + \frac{10}{25} - 6B \Rightarrow 1 = \frac{19}{25} - 6B$

$= \frac{9 + 28}{25} - 6B$ $\quad 1 - \frac{19}{25} = -6B$

$1 = \frac{17}{25} - 6B$ $\quad \frac{6}{25} = -6B$

$1 - \frac{29}{25} = -6B$ $\quad \boxed{B = -\frac{1}{25}}$

$-\frac{4}{25} =$

$$\frac{F(z)}{z} = \frac{\frac{1}{25}}{z-2} + \frac{-\frac{1}{25}}{z+3} + \frac{(-\frac{1}{5})}{(z+3)^2}$$

$$\frac{F(z)}{z} = \frac{1}{25}\frac{1}{z-2} - \frac{1}{25}\frac{1}{z+3} - \frac{1}{5}\frac{1}{(z+3)^2}$$

$$= \frac{1}{25}\frac{z}{z-2} - \frac{1}{25}\frac{z}{z+3} - \frac{1}{5}\frac{z}{(z+3)^2}$$

$$Z[y(n)] = \frac{1}{25}\frac{z}{z-2} - \frac{1}{25}\frac{z}{z+3} - \frac{1}{5.3}\frac{3z}{(z+3)^2}$$

$$y(n) = \frac{1}{25}Z^{-1}\left[\frac{z}{z-2}\right] - \frac{1}{25}Z^{-1}\left[\frac{z}{z+3}\right] - \frac{1}{15}Z^{-1}\left[\frac{3z}{(z+3)^2}\right]$$

$= \frac{1}{25}[2^n] - \frac{1}{25}[(-3)]^n + \frac{1}{15} n (-3)^n$

$= \frac{1}{25}\left[2^n - (-3)^n - \frac{5}{3} n (-3)^n\right]$

2) $y(n+3) - 3y(n+1) + 2y(n) = 0$

$y(0) = 4, \quad y(1) = 0 \quad \& \quad y(2) = 8$

Solution:

$Z[y(n+3)] - 3Z\, y(n+1) + 2Z(y(n)) = 0$

$z^3 F[z] - z^3 y(0) - z^2 y(1) - z y(2) - 3[z F[z] - z y[0]] + 2[F[z]] = 0$

$[z^3 F[z] - 4z^3 - 0 - 8z] - 3[z F(z) - 4z] + 2 F[z] = 0$

$z^3 F[z] - 4z^3 - 8z - 3z F[z] + 12z + 2F[z] = 0$

$z^3 F[z] - 3z F[z] + 2F[z] = 4z^3 + 8z - 12z$

$F[z][z^3 - 3z + 2] = 4z^3 - 4z$

$F[z] = \frac{4z^3 - 4z}{z^3 - 3z + 2} = \frac{4z^3 - 4z}{(z-1)^2 (z+2)}$

$F[z] = \frac{4z^3 - 4z}{(z-1)^2 (z+3)}$

$z^3 - 3z + 2$

$$\begin{array}{r|rrrr} 1 & 1 & 0 & -3 & 2 \\ & 0 & 1 & 1 & -2 \\ \hline & 1 & 1 & -2 & \boxed{0} \end{array}$$

$(x-1)(x^2 + x - 2) = 0$

$(x-1)(x-1)(x+2) = 0$

$(x-1)^2 (x+2) = 0$

$F[z]\, z^{n-1} = \frac{z(4z^2 - 4) z^{n-1}}{(z-1)^2 (z+3)}$

$F[z]\, z^{n-1} = \frac{[4z^2 - 4]\, z^n}{(z-1)^2 (z+3)}$

$= \frac{4(z+1)\cancel{(z-1)}\, z^n}{(z-1)^{\cancel{2}} (z+2)}$

$= 4(z+1) z^n$

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$$F[z]\, z^{n-1} = \frac{4(z+1)\, z^n}{(z-1)(z+2)}$$

$z = 1$ is a simple pole

$z = -2$ is a simple pole

$$\underset{\text{at } z=1}{\text{Res}} = \lt_{z \to 1} \cancel{(z-1)} \frac{4(z+1)\, z^n}{\cancel{(z-1)}(z+2)}$$

$$= \lt_{z \to 1} \frac{4(z+1)\, z^n}{z+2}$$

$$= 8/3$$

$$\underset{\text{at } z=-2}{\text{Res}} = \lt_{z \to -2} \cancel{(z+2)} \frac{4(z+1)\, z^n}{(z-1)\cancel{(z+2)}}$$

$$= \lt_{z \to -2} \frac{4(z+1)\, z^n}{(z-1)}$$

$$= \frac{-4(-2)^n}{-3} = 4/3\, (-2)^n$$

$y(n)$ = sum of the residues

$$= 8/3 + 4/3\, (-2)^n$$