TOPIC: 3.2 – PARTIAL DERIVATIVES

A partial derivative is the rate at which a multivariable function changes with respect to one of its variables, while keeping all other variables constant. For a function $f(x, y, z, \ldots)$, the partial derivative with respect to x is denoted as $\frac{\partial f}{\partial x}$ and represents how f changes as x varies, treating $y, z, \ldots$ as constants. It is a key concept in calculus used to analyze functions with multiple inputs, with applications in optimization, physics, engineering, and machine learning.

Notation

The partial derivative of f with respect to x is commonly written as:

- $\frac{\partial f}{\partial x}$
- f_x
- $\partial_x f$

Similarly, for the other variables like y or z, we use $\frac{\partial f}{\partial y}$ or $\frac{\partial f}{\partial z}$.

Applications

Partial derivatives are used in:

- Optimization (finding maxima and minima of functions)
- Differential equations
- Physics and engineering (heat, wave, and fluid dynamics)
- Machine learning and data science (gradient-based algorithms)

Problems based on Partial derivatives

1. If $u = (x-y)(y-z)(z-x)$, then show that

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0.$$

Given $u = (x-y)(y-z)(z-x)$

$$\frac{\partial u}{\partial x} = (y-z)\left[(x-y)(-1) + (z-x)(1)\right]$$

$$= -(x-y)(y-z) + (y-z)(z-x)$$

$$\frac{\partial u}{\partial y} = (z-x)\left[(x-y)(1) + (y-z)(-1)\right]$$

$$\frac{\partial u}{\partial y} = (z-x)\left[(x-y)(1) + (y-z)(-1)\right]$$

$$= (x-y)(z-x) - (y-z)(z-x)$$

$$\frac{\partial u}{\partial z} = (x-y)\left[(y-z)(+1) + (z-x)(-1)\right]$$

$$= (x-y)(y-z) - (x-y)(z-x)$$

$$\therefore \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$$

Euler's Theorem for homogeneous function

If u is a homogeneous function of degree 'n' in x and y, then

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu$$

(1) If $u = \sin^{-1}\left[\frac{x^3 - y^3}{x+y}\right]$, then prove that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2\tan u$.

Let $f(x,y) = \sin u = \frac{x^3 - y^3}{x+y}$

$$f(tx, ty) = \frac{t^3x^3 - t^3y^3}{tx+ty} = t^2 f(x,y)$$

$\therefore$ f is a homogeneous function of degree 2 in x & y.

$\therefore$ By Euler's theorem, we get

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = nf = 2f \rightarrow (1)$$

Here $f = \sin u$

sub. in (1).

$$x\frac{\partial}{\partial x}(\sin u) + y\frac{\partial}{\partial y}(\sin u) = 2\sin u$$

$$x\left[\cos u\frac{\partial u}{\partial x}\right] + y\left[\cos u\frac{\partial u}{\partial y}\right] = 2\sin u.$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2\tan u.$$

(2) If $u = \frac{x}{y} + \frac{y}{x} + \frac{z}{x}$, then find $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z}$.

Given $u(x, y, z) = \frac{x}{y} + \frac{y}{z} + \frac{z}{x}$

$$u(tx, ty, tz) = \frac{tx}{ty} + \frac{ty}{tz} + \frac{tz}{tx}$$

$$= t^0 \, u(x, y, z)$$

$\therefore$ u is a homogeneous function of x, y, z in degree 0.

$\therefore$ By Euler's theorem,

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = nu$$

$$= 0.$$

(4) If $u = \cos^{-1}\left[\frac{x+y}{\sqrt{x}+\sqrt{y}}\right]$, then prove that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = -\frac{1}{2}\cot u$.

Let $f(x, y) = \cos u = \frac{x+y}{\sqrt{x}+\sqrt{y}}$

$$f(tx, ty) = \frac{tx + ty}{\sqrt{tx} + \sqrt{ty}} = t^{1/2} f(x, y)$$

$\Rightarrow$ f is a homogeneous function of degree $\frac{1}{2}$ in x and y.

∴ By Euler's theorem.

$$x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y} = nf = \frac{1}{2} f \rightarrow (1)$$

Here $f = \cos u$, sub in (1).

$$x \frac{\partial}{\partial x}(\cos u) + y \frac{\partial}{\partial y}(\cos u) = \frac{1}{2} \cos u$$

$$x \left[-\sin u \frac{\partial u}{\partial x}\right] + y \left[-\sin u \frac{\partial u}{\partial y}\right] = \frac{1}{2} \cos u$$

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = -\frac{1}{2} \frac{\cos u}{\sin u}$$

$$= -\frac{1}{2} \cot u$$

(7) If $u = (x^2+y^2+z^2)^{-1/2}$, then find the value of $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}$.

Given $u = (x^2+y^2+z^2)^{-1/2}$

$$\frac{\partial u}{\partial x} = -\frac{1}{2}(x^2+y^2+z^2)^{-3/2}(2x)$$

$$\frac{\partial u}{\partial x} = -x(x^2+y^2+z^2)^{-3/2}$$

$$\frac{\partial^2 u}{\partial x^2} = -\left[x\left(-\frac{3}{2}\right)(x^2+y^2+z^2)^{-5/2}(2x) + (x^2+y^2+z^2)^{-3/2}\right]$$

$$\frac{\partial^2 u}{\partial x^2} = 3x^2(x^2+y^2+z^2)^{-5/2} - (x^2+y^2+z^2)^{-3/2} \rightarrow (1)$$

|||ly

$$\frac{\partial^2 u}{\partial y^2} = 3y^2(x^2+y^2+z^2)^{-5/2} - (x^2+y^2+z^2)^{-3/2} \longrightarrow (2)$$

$$\frac{\partial^2 u}{\partial z^2} = 3z^2(x^2+y^2+z^2)^{-5/2} - (x^2+y^2+z^2)^{-3/2} \longrightarrow (3)$$

(1) + (2) + (3)

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}$$

$$= (x^2+y^2+z^2)^{-5/2}\left[3x^2+3y^2+3z^2\right] - 3(x^2+y^2+z^2)^{-3/2}$$

|||ly

$$\frac{\partial^2 u}{\partial y^2} = 3y^2(x^2+y^2+z^2)^{-5/2} - (x^2+y^2+z^2)^{-3/2} \longrightarrow (2)$$

$$\frac{\partial^2 u}{\partial z^2} = 3z^2(x^2+y^2+z^2)^{-5/2} - (x^2+y^2+z^2)^{-3/2} \longrightarrow (3)$$

(1) + (2) + (3)

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}$$

$$= (x^2+y^2+z^2)^{-5/2}\left[3x^2+3y^2+3z^2\right] - 3(x^2+y^2+z^2)^{-3/2}$$

$$= 3(x^2+y^2+z^2)(x^2+y^2+z^2)^{-5/2} - 3(x^2+y^2+z^2)^{-3/2}$$

$$= 3(x^2+y^2+z^2)^{-3/2} - 3(x^2+y^2+z^2)^{-3/2}$$

$$= 0$$