TOPIC: 3.3 TOTAL DERIVATIVES

Total Derivatives

If $u = f(x,y)$, where $x = \phi(t)$ and $y = \psi(t)$ then we can express u as a function of t alone by substituting the values of x and y in $f(x,y)$. Thus, we can find the ordinary derivative $\frac{du}{dt}$ which is called the total derivative of u to distinguish it from the partial derivatives $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$. i) $\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt}$

Composite function of one variable

If $u = f(x,y,z)$ where x, y, z are all functions of a variable t, then

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt} + \frac{\partial u}{\partial z}\frac{dz}{dt}$$

Differentiation of Implicit functions

If $f(x, y) = c$ be an implicit relation between x and y which defines as a differentiable function of x, then

$$\frac{dy}{dx} = \frac{-\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}} \qquad \left[\because \frac{\partial f}{\partial y} \neq 0\right]$$

Compositive function of two variables

If $z = f(x, y)$ where $x = \phi(u, v)$, $y = \psi(u, v)$, then z is a function of u, v

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u}$$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v}.$$

(1) Find $\frac{dy}{dx}$ when $x^3 + y^3 = 3axy$.

Let $f(x, y) = x^3 + y^3 - 3axy$

$$\frac{dy}{dx} = -\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}} = -\frac{3x^2 - 3ay}{3y^2 - 3ax}$$

$$= -\frac{x^2 - ay}{y^2 - ax}$$

(5) If $Z = f(y-z, z-x, x-y)$, show that $\frac{\partial Z}{\partial x} + \frac{\partial Z}{\partial y} + \frac{\partial Z}{\partial z} = 0$.

Let $u = y - z$, $v = z - x$, $w = x - y$

$$Z = f(u, v, w)$$

$$\frac{\partial Z}{\partial x} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial x} + \frac{\partial f}{\partial w}\frac{\partial w}{\partial x}$$

$$= \frac{\partial f}{\partial u}(0) + \frac{\partial f}{\partial v}(-1) + \frac{\partial f}{\partial w}(1)$$

$$= -\frac{\partial f}{\partial v} + \frac{\partial f}{\partial w}$$

Illy $\frac{\partial Z}{\partial y} = +\frac{\partial f}{\partial u} \mp \frac{\partial f}{\partial w}$

(11) If $u = \log(x^2+y^2) + \tan^{-1}(y/x)$, prove that $u_{xx} + u_{yy} = 0$.

$$u_x = \frac{\partial u}{\partial x} = \frac{1}{x^2+y^2}(2x) + \frac{1}{1+(y/x)^2}\left[-\frac{y}{x^2}\right]$$

$$= \frac{2x}{x^2+y^2} + \frac{1}{\frac{x^2+y^2}{x^2}}\left[-\frac{y}{x^2}\right]$$

$$= \frac{2x}{x^2+y^2} - \frac{y}{x^2+y^2} = \frac{2x-y}{x^2+y^2}$$

$$u_{xx} = \frac{\partial^2 u}{\partial x^2} = \frac{(x^2+y^2)(2) - (2x-y)(2x)}{(x^2+y^2)^2}$$

$$= \frac{2x^2+2y^2-4x^2+2xy}{(x^2+y^2)^2}$$

$$u_{xx} = \frac{2y^2 - 2x^2 + 2xy}{(x^2+y^2)^2} \longrightarrow (1)$$

$$u_y = \frac{\partial u}{\partial y} = \frac{1}{x^2+y^2}(2y) + \frac{1}{1+\left(\frac{y}{x}\right)^2}\left(\frac{1}{x}\right)$$

$$= \frac{2y}{x^2+y^2} + \frac{1}{\frac{x^2+y^2}{x^2}}\left(\frac{1}{x}\right)$$

$$= \frac{2y}{x^2+y^2} + \frac{x}{x^2+y^2} = \frac{2y+x}{x^2+y^2}$$

$$u_{yy} = \frac{(x^2+y^2)(2) - (2y+x)(2y)}{(x^2+y^2)^2}$$

$$= \frac{2x^2+2y^2 - 4y^2 - 2xy}{(x^2+y^2)^2}$$

$$u_{yy} = \frac{2x^2 - 2y^2 - 2xy}{(x^2+y^2)^2} \longrightarrow (2)$$

(1) + (2)

$$u_{xx} + u_{yy} = \frac{2y^2 - 2x^2 + 2xy + 2x^2 - 2y^2 - 2xy}{(x^2+y^2)^2}$$

$$= 0$$