TOPIC: 3.5 - JACOBIANS AND PROPERTIES

Jacobians

If $u_1, u_2, \cdots, u_n$ are functions of n variables $x_1, x_2, \cdots, x_n$, then the Jacobian of the transformation from $x_1, x_2, \cdots x_n$ to $u_1, u_2, \cdots, u_n$ is defined by

$$\begin{vmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} & \cdots & \frac{\partial u_1}{\partial x_n} \\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} & \cdots & \frac{\partial u_2}{\partial x_n} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial u_n}{\partial x_1} & \frac{\partial u_n}{\partial x_2} & \cdots & \frac{\partial u_n}{\partial x_n} \end{vmatrix}$$

and is denoted by the symbol $\frac{\partial(u_1, u_2, \cdots u_n)}{\partial(x_1, x_2, \cdots x_n)}$ or $J_{(u_1, u_2, \cdots u_n)}$.

In particular $\frac{\partial(u_1, u_2)}{\partial(x_1, x_2)} = \begin{vmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} \\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} \end{vmatrix}$

$$\frac{\partial(u_1, u_2, u_3)}{\partial(x_1, x_2, x_3)} = \begin{vmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} & \frac{\partial u_1}{\partial x_3} \\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} & \frac{\partial u_2}{\partial x_3} \\ \frac{\partial u_3}{\partial x_1} & \frac{\partial u_3}{\partial x_2} & \frac{\partial u_3}{\partial x_3} \end{vmatrix}$$

Functional dependence

If u, v, w are functionally dependent functions of the independent variables x, y, z then $\frac{\partial(u,v,w)}{\partial(x,y,z)} = 0$.

(1) If $x = r\cos\theta$, $y = r\sin\theta$, find (i) $\frac{\partial(x,y)}{\partial(r,\theta)}$ (ii) $\frac{\partial(r,\theta)}{\partial(x,y)}$.

Given $x = r\cos\theta$ $\quad$ $y = r\sin\theta$

$$\frac{\partial x}{\partial r} = \cos\theta \qquad \frac{\partial y}{\partial r} = \sin\theta$$

$$\frac{\partial x}{\partial \theta} = -r\sin\theta \qquad \frac{\partial y}{\partial \theta} = r\cos\theta$$

(i) $$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix}$$

$$= \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

$$= r\cos^2\theta + r\sin^2\theta = r$$

(ii) $$\frac{\partial(r,\theta)}{\partial(x,y)} = \frac{1}{r}$$

(6) Find the Jacobian of y_1, y_2, y_3 w.r.t x_1, x_2, x_3 if $y_1 = \frac{x_2 x_3}{x_1}$, $y_2 = \frac{x_3 x_1}{x_2}$, $y_3 = \frac{x_1 x_2}{x_3}$.

$$\frac{\partial(y_1, y_2, y_3)}{\partial(x_1, x_2, x_3)} = \begin{vmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} & \frac{\partial y_1}{\partial x_3} \\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} & \frac{\partial u_2}{\partial x_3} \\ \frac{\partial u_3}{\partial x_1} & \frac{\partial u_3}{\partial x_2} & \frac{\partial u_3}{\partial x_3} \end{vmatrix}$$

$$= \begin{vmatrix} -\frac{x_2 x_3}{x_1^2} & \frac{x_3}{x_1} & \frac{x_2}{x_1} \\ \frac{x_3}{x_2} & -\frac{x_3 x_1}{x_2^2} & \frac{x_1}{x_2} \\ \frac{x_2}{x_3} & \frac{x_1}{x_3} & -\frac{x_1 x_2}{x_3^2} \end{vmatrix}$$

$$= -\frac{x_2 x_3}{x_1^2}\left[\frac{x_1^2 x_2 x_3}{x_2^2 x_3^2} - \frac{x_1^2}{x_2 x_3}\right] - \frac{x_3}{x_1}\left[-\frac{x_1 x_2 x_3}{x_2 x_3^2} - \frac{x_1 \ldots}{x_2 \ldots}\right.$$

$$+ \frac{x_2}{x_1}\left[\frac{x_1 x_3}{x_2 x_3} + \frac{x_1 x_2 x_3}{x_2^2 x_3}\right]$$

$$= -1 + 1 + 1 + 1 + 1 + 1$$

$$= 4$$