TOPIC:3.7- MAXIMA AND MINIMA OF FUNCTIONS OF TWO VARIABLES

(a)Maximum value

$f(a,b)$ is a maximum value of $f(x,y)$, if there exists some neighbourhood of the point (a, b) such that for every point (a + h ,b +k) of the neighbourhood.

$$f(a,b) > f(a+h, b+k)$$

(b) Minimum value

$f(a,b)$ is a minimum value of $f(x,y)$,if there exists some neighborhood of the point (a, b) such that for every point (a + h ,b +k) of the neighborhood.

$$f(a,b) < f(a+h, b+k)$$

(c) Extremum value

$f(a,b)$ is said to be an extremum value of $f(x,y)$ if it is either a maximum or minimum.

(d) Necessary conditions for a maximum or a minimum.

$$f_x(a,b) = 0 \ \ and \ \ f_y(a,b) = 0$$

Notations : $f_x = \frac{\partial f}{\partial x},\ f_y = \frac{\partial f}{\partial y},\ f_{xx} = \frac{\partial^2 f}{\partial x^2},\ f_{xy} = \frac{\partial^2 f}{\partial x \partial y},\ f_{yy} = \frac{\partial^2 f}{\partial y^2}$

(e) Sufficient conditions:

If $f_x(a,b) = 0 \ and \ f_y(a,b) = 0$ and $f_{xx}(a,b) = A,\ f_{xy}(a,b) = B$, $f_{yy}(a,b) = C$, then

(i) $f(a,b)$ is maximum value if $AC - B^2 > 0 \ \ and \ A < 0 \ or \ B < 0$

(ii) $f(a,b)$ is minimum value if $AC - B^2 > 0 \ \ and \ A > 0 \ or \ B > 0$

(iii) $f(a,b)$ is not an extremum (saddle) if $AC - B^2 < 0$

(iv) if $AC - B^2 = 0$ then the test is inconclusive.

(f) Stationary value

A function $f(x,y)$ is said to be stationary at (a, b) or $f(a,b)$ is said to be

Stationary value of $f(x,y)$ if $f_x(a,b) = 0$ *and* $f_y(a,b) = 0$

Note:

Every extremum value is a stationary value but a stationary value need not be an extremum value

Problems Based on Maxima and Minima for Functions of Two Variables

Example:

Find the extreme values of the function $f(x,y) = x^3 + y^3 - 3x - 12y + 20$

Solution:

Given $f(x,y) = x^3 + y^3 - 3x - 12y + 20$

$f_x = 3x^2 - 3$; $f_y = 3y^2 - 12$

$f_{xx} = 6x = A$, $f_{xy} = 0 = B$, $f_{yy} = 6y = C$

To find the stationary points.

$f_x = 0$	$f_x = 0$
$3x^2 - 3 = 0$ $x^2 - 1 = 0$ $x = \pm 1$	$3y^2 - 12 = 0$ $y^2 - 4 = 0$ $y = \pm 2$

$\therefore$ Stationary points are $(1,2), (1,-2), (-1,2), (-1,-2)$

	$(1,2)$	$(1,-2)$	$(-1,2)$	$(-1,-2)$
$A = 6x$	$6 > 0$	$6 > 0$	$-6 < 0$	$-6 < 0$
$B = 0$	0	0	0	0
$C = 6y$	12	-12	12	-12
$AC - B^2$	$72 > 0$	$-72 < 0$	$-72 < 0$	$72 > 0$
Conclusion	Min. point	Saddle point	Saddle point	Max. point

$\therefore$ Maximum value of $f(x,y)$ is

$$f(-1,-2) = (-1)^3 + (-2)^3 - 3(-1) - 12(-2) + 20$$
$$= -1 - 8 + 3 + 24 + 20 = 38$$

Minimum value of $f(x,y)$ is

$$f(1,2) = (1)^3 + (2)^3 - 3(1) - 12(2) + 20 = 2$$

Example:

A flat circular plate is heated so that the temperature at any point (x,y) is $u(x,y) = x^2 + 2y^2 - x$. Find the coldest point on the plate.

Solution:

$u(x,y) = x^2 + 2y^2 - x$

$u_x = 2x - 1 \qquad u_y = 4y$

$u_x = 0$ $\Rightarrow 2x - 1 = 0$ $\Rightarrow x = \frac{1}{2}$	$u_y = 0$ $4y = 0$ $y = 0$

$A = u_{xx} = 2$; $C = u_{yy} = 4 \quad B = u_{xy} = 0$

$\Delta = AC - B^2 > 0$

U is minimum at $\left(\frac{1}{2}, 0\right)$ and its minimum value is $-\frac{1}{4}$

Example:

Find the maxima and minima of $x^4 + y^4 - 2x^2 + 4xy - 2y^2$

Solution:

$f(x,y) = x^4 + y^4 - 2x^2 + 4xy - 2y^2$

$f_x = 4x^3 - 4x + 4y \qquad ; \; f_y = 4y^3 + 4x - 4y$

$f_{xx} = 12x^2 - 4 = A$, $f_{xy} = 4 = B$, $f_{yy} = 12y^2 - 4 = C$

To find the stationary points.

$f_x = 0$	$f_y = 0$

$4x^3 - 4x + 4y = 0$ $x^3 - x + y = 0 \ldots \ (1)$	$4y^3 + 4x - 4y = 0$ $y^3 + x - y = 0 \ldots \ (2)$

$$(1) + (2) \Rightarrow x^3 + y^3 = 0 \quad \Rightarrow x^3 = -y^3 \quad \Rightarrow y = -x$$

$$(1) \Rightarrow x^3 - x - x = 0 \quad \Rightarrow x^3 - 2x = 0 \Rightarrow x(x^2 - 2) = 0$$

$$\Rightarrow x = 0 \ (or) \ (x^2 - 2) = 0$$

$$\Rightarrow x = 0 \ (or) \ x = \pm\sqrt{2}$$

$\therefore$ The stationary points are $(0,0), (\sqrt{2}, -\sqrt{2}), (-\sqrt{2}, \sqrt{2})$

	$(0,0)$	$(\sqrt{2}, -\sqrt{2})$	$(-\sqrt{2}, \sqrt{2})$
$A = 12x^2 - 4$	$-4 < 0$	$20 > 0$	$20 > 0$
$B = 4$	4	4	4
$C = 12y^2 - 4$	-4	20	20
$AC - B^2$	0	$384 > 0$	$384 > 0$
Conclusion	Cannot be an extreme point	Minimum point	Minimum point

Minimum at $(\sqrt{2}, -\sqrt{2})$

$$= (\sqrt{2})^4 + (-\sqrt{2})^4 - 2(\sqrt{2})^2 + 4\sqrt{2}(-\sqrt{2}) - 2(-\sqrt{2})^2$$

$$= 4 + 4 - 4 - 8 - 4$$

$$= -8$$

Minimum at $(-\sqrt{2}, \sqrt{2})$

$$= (-\sqrt{2})^4 + (\sqrt{2})^4 - 2(-\sqrt{2})^2 + 4(-\sqrt{2})\sqrt{2} - 2(\sqrt{2})^2$$

$$= 4 + 4 - 8 - 4 - 4 = -8$$