TOPIC:3.11- PROBLEMS ON LAGRANGES

1) A rectangular box open at the top, is to have a volume of 32 cc. Find the dimensions of the box, that requires least materials for its construction.

Let x, y, z be the dimensions of the rectangular box.

Surface area $f(x,y,z) = xy + 2xz + 2yz$

volume $g(x,y,z) = xyz = 32$

$g(x,y,z) = xyz - 32$

Hence $F(x,y,z) = f(x,y,z) + \lambda g(x,y,z)$

$$= xy + 2xz + 2yz + \lambda(xyz - 32) \longrightarrow (1)$$

Diff. (1) partially w.r.t 'x', 'y', 'z'.

$$\frac{\partial F}{\partial x} = y + 2z + yz\lambda = 0$$

$$yz\lambda = -y - 2z$$

$$\lambda = -\left[\frac{y+2z}{yz}\right]$$

$$\lambda = -\left[\frac{1}{z} + \frac{2}{y}\right] \rightarrow (2)$$

$$\frac{\partial F}{\partial y} = x + 2z + xz\lambda = 0$$

$$xz\lambda = -x - 2z$$

$$\lambda = -\left[\frac{x+2z}{xz}\right]$$

$$\lambda = -\left[\frac{1}{z} + \frac{2}{x}\right] \rightarrow (3)$$

$$\frac{\partial F}{\partial z} = 2x + 2y + xy\lambda = 0$$

$$xy\lambda = -2x - 2y$$

$$\lambda = -\left[\frac{2x+2y}{xy}\right]$$

$$\lambda = -\left[\frac{2}{y} + \frac{2}{x}\right] \longrightarrow (4)$$

From (2) & (3)

$$-\left[\frac{1}{z} + \frac{2}{y}\right] = -\left[\frac{1}{z} + \frac{2}{x}\right]$$

$$\frac{2}{y} = \frac{2}{x} \Rightarrow \boxed{x = y}$$

From (3) & (4),

$$-\left[\frac{1}{z} + \frac{2}{x}\right] = -\left[\frac{2}{y} + \frac{2}{x}\right]$$

$$\frac{1}{z} = \frac{2}{y} \Rightarrow \boxed{y = 2z}$$

$$\therefore \quad x = y = 2z$$

Sub. in $g(x, y, z)$.

$$xyz = 32 \Rightarrow 2z \cdot 2z \cdot z = 32$$

$$\Rightarrow 4z^3 = 32 \Rightarrow z^3 = 8 \Rightarrow \boxed{z = 2}$$

$\boxed{x = 4}$ and $\boxed{y = 4}$

2) Find the dimensions of the rectangular box, open at the top, of maximum capacity whose surface area is 432 squar meter.

Let x, y, z be the rectangular box dimension

volume $f(x,y,z) = xyz$

surface area $g(x,y,z) = xy + 2yz + 2xz = 432$

$$g(x,y,z) = xy + 2yz + 2xz - 432$$

Hence $F(x,y,z) = f(x,y,z) + \lambda\, g(x,y,z)$

$$= xyz + \lambda(xy + 2yz + 2xz - 432) \longrightarrow (1)$$

Diff. (1) partially w.r.t 'x', 'y' and 'z'

$$\frac{\partial F}{\partial x} = yz + \lambda(y+2z) = 0$$

$$\lambda(y+2z) = -yz$$

$$\lambda = \frac{-yz}{y+2z} \rightarrow (2)$$

$$\frac{\partial F}{\partial y} = xz + \lambda(x+2z) = 0$$

$$\lambda(x+2z) = -xz$$

$$\lambda = \frac{-xz}{x+2z} \rightarrow (3)$$

$$\frac{\partial F}{\partial z} = xy + \lambda(2y+2x) = 0$$

$$\lambda(2y+2x) = -xy$$

$$\lambda = -\frac{xy}{2y+2x} \rightarrow (4)$$

From (2) & (3),

$$\frac{-yz}{y+2z} = \frac{-xz}{x+2z} \Rightarrow \frac{y}{y+2z} = \frac{x}{x+2z}$$

$$\Rightarrow xy + 2yz = xy + 2xz$$

$$\Rightarrow 2yz = 2xz \Rightarrow \boxed{x = y}$$

From ③ & ④,

$$-\frac{xz}{x+2z} = -\frac{xy}{2y+2x} \Rightarrow \frac{z}{x+2z} = \frac{y}{2y+}$$

$$\Rightarrow 2yz + 2xz = xy + 2yz$$

$$\Rightarrow 2xz = xy \Rightarrow \boxed{y = 2z}$$

$$\therefore \boxed{x = y = 2z}$$

sub. in $g(x, y, z)$

$$xy + 2yz + 2xz = 432$$

sub. in $g(x, y, z)$

$$xy + 2yz + 2xz = 432$$

$$(2z)(2z) + 2(2z)z + 2(2z)z = 432$$

$$\Rightarrow 4z^2 + 4z^2 + 4z^2 = 432$$

$$\Rightarrow 12z^2 = 432 \Rightarrow z^2 = 36$$

$$\Rightarrow \boxed{z = 6}$$

$$\therefore \boxed{x = 12} \quad \boxed{y = 12}$$