TOPIC 3: SOLUTIONS OF ONE DIMENSIONAL WAVE EQUATION

A tightly stretched string with fixed end points $x=0$ & $x=l$ is initially displaced in the position $y=y_0 \sin^3\left(\frac{\pi x}{l}\right)$ and then released from rest. Find the displacement y at any distance x from one end at time t.

Soln:

$$\overbrace{\qquad\qquad\qquad\qquad}^{l}$$
$x=0 \qquad\qquad x=l$

The displacement $y(x,t)$ is from $a^2 \frac{\partial^2 y}{\partial x^2} = \frac{\partial^2 y}{\partial t^2}$

The Conditions are

(i) $y(0,t)=0$

(ii) $y(l,t)=0$

(iii) $\left(\frac{\partial y}{\partial t}\right)_{t=0}=0$

(iv) $y(x,0) = y_0 \sin^3\left(\frac{\pi x}{l}\right)$

$\sin^3\theta = \frac{1}{4}\left[3\sin\theta - \sin 3\theta\right]$

$$= \frac{y_0}{4}\left[3\sin\left(\frac{\pi x}{l}\right) - \sin\left(\frac{3\pi x}{l}\right)\right]$$

The Suitable soln. is

$$y(x,t) = (c_1 \cos px + c_2 \sin px)(c_3 \cos pat + c_4 \sin pat) \quad \text{--- (1)}$$

Apply (i) $y(0,t)=0$

$$c_1 (c_3 \cos pat + c_4 \sin pat) = 0$$

$$\therefore \boxed{c_1 = 0}$$

Apply (ii) in (1) $x=l$

$$y(l,t) = c_2 \sin pl\,(c_3 \cos pat + c_4 \sin pat)$$

$$c_2 \sin pl = 0$$

$$\sin pl = 0 \Rightarrow pl = n\pi \Rightarrow p = n\pi/l.$$

(1) Becomes

$$y(x,t) = B_2 \sin\frac{n\pi x}{l}\left(c_3 \cos\frac{n\pi at}{l} + c_4 \sin\frac{n\pi at}{l}\right) \quad \text{--- (2)}$$

The most general soln is

$$y(x,t) = \cancel{\sum_{n=1}^{\infty}}$$

now apply cond (iii) we get

$$\left(\frac{\partial y}{\partial t}\right)_{(x,0)} = C_2 \sin\left(\frac{n\pi x}{l}\right) C_4 \frac{n\pi a}{l} = 0$$

$$C_4 = 0$$

$$y(x,t) = C_n \sin\left(\frac{n\pi x}{l}\right) \cos\left(\frac{n\pi at}{l}\right)$$

The most general soln

$$y(x,t) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi x}{l}\right) \cos\left(\frac{n\pi at}{l}\right)$$

Apply (iv) $y(x,0) = f(x)$

$$\frac{3y_0}{4} \sin\left(\frac{\pi x}{l}\right) - \frac{y_0}{4} \sin\left(\frac{3\pi x}{l}\right) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi x}{l}\right)$$

$$= C_1 \sin\left(\frac{\pi x}{l}\right) + C_2 \sin\left(\frac{2\pi x}{l}\right) + C_3 \sin\left(\frac{3\pi x}{l}\right) + \cdots$$

On Comparing

$$\frac{3y_0}{4} = C_1 \quad C_2 = 0 \quad C_3 = -\frac{y_0}{4}$$

$$C_4 = C_5 = \cdots = 0$$

$$y(x,t) = \frac{3y_0}{4} \sin\left(\frac{\pi x}{l}\right) \cos\left(\frac{\pi at}{l}\right) - \frac{y_0}{4} \sin\left(\frac{3\pi x}{l}\right) \cos\left(\frac{3\pi at}{l}\right)$$

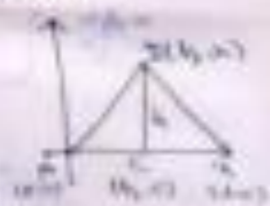
A tightly stretched string of length l has its ends fastened at $x=0$ and $x=l$. The mid point of the string is then drawn to a height h and then released from rest in that position. Obtain an expression for the displacement of the string at any subsequent time.

Soln: The eqn. to be solved is $\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2}$

From the given problem, we get the following boundary and initial conditions

(1) $y(0,t)=0$ (2) $y(l,t)=0$ (3) $\left(\frac{\partial y}{\partial t}\right)_{t=0}=0$

Eqn. of AB is



$$\frac{x-x_1}{x_2-x_1} = \frac{y-y_1}{y_2-y_1} \Rightarrow \frac{x-0}{l/2-0} = \frac{y-0}{h-0}$$

$$xh = \frac{l}{2} y \Rightarrow y = \frac{2}{l} xh$$

Eqn. of BC is

$$\frac{x-l/2}{l-l/2} = \frac{y-h}{0-h} \Rightarrow \frac{x-l/2}{l/2} = \frac{y-h}{-h}$$

$$y-h = -\frac{2h}{l}(x-l/2) \Rightarrow y = h - \frac{2hx}{l} + h$$

$$y = 2h(1-x/l) = \frac{2h}{l}(l-x)$$

(4) $y(x,0) = \begin{cases} \frac{2hx}{l} & 0<x<l/2 \\ \frac{2h}{l}(l-x) & l/2<x<l \end{cases}$

(5) $y(x,t) = \sum_{n=1}^{\infty} C_n \sin\frac{n\pi x}{l}\cos\frac{n\pi a t}{l}$

Applying cond. (iv) in (5)

$$y(x,0) = \sum_{n=1}^{\infty} C_n \sin\frac{n\pi x}{l} = \begin{cases} \frac{2hx}{l} & 0<x<l/2 \\ \frac{2h(l-x)}{l} & l/2<x<l. \end{cases}$$

To find C_n :- Expand the value in a half-range sine series

$$\left.\begin{array}{l} \frac{2hx}{l}, \quad 0<x<l/2 \\ \frac{2h}{l}(l-x), \quad l/2<x<l \end{array}\right\} = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

$$b_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\,dx$$

$$C_n = \frac{2}{l}\left[\int_0^{l/2}\frac{2hx}{l}\sin\frac{n\pi x}{l}\,dx + \int_{l/2}^{l}\frac{2h(l-x)}{l}\sin\frac{n\pi x}{l}\,dx\right]$$

$$= \frac{4h}{l^2}\left[\int_0^{l/2} x\sin\frac{n\pi x}{l}\,dx + \int_{l/2}^{l}(l-x)\sin\frac{n\pi x}{l}\,dx\right]$$

$$= \frac{4h}{l^2}\left[-x\frac{l}{n\pi}\cos\frac{n\pi x}{l} + \left(\frac{l}{n\pi}\right)^2\sin\frac{n\pi x}{l}\right]_0^{l/2}$$

$$+ \frac{4h}{l^2}\left[-(l-x)\frac{l}{n\pi}\cos\frac{n\pi x}{l} - \left(\frac{l}{n\pi}\right)^2\sin\frac{n\pi x}{l}\right]_{l/2}^{l}$$

$$= \frac{4h}{l^2}\left[\left(-\frac{l^2}{2n\pi}\cos\frac{n\pi}{2} + \left(\frac{l}{n\pi}\right)^2\sin\frac{n\pi}{2}\right) - (0+0)\right]$$

$$+ \frac{4h}{l^2}\left[(-0-0) - \left(-\frac{l^2}{2n\pi}\cos\frac{n\pi}{2} - \left(\frac{l}{n\pi}\right)^2\sin\frac{n\pi}{2}\right)\right]$$

$$= \frac{4h}{l^2}\left[-\frac{l^2}{2n\pi}\cos\frac{n\pi}{2} + \left(\frac{l}{n\pi}\right)^2\sin\frac{n\pi}{2} + \frac{l^2}{2n\pi}\cos\frac{n\pi}{2} + \left(\frac{l}{n\pi}\right)^2\sin\frac{n\pi}{2}\right]$$

$$\therefore C_n = \frac{b}{l^2}\left[\int_0^l x \sin\frac{n\pi x}{2l}\,dx + \int_l^{2l}(2l-x)\sin\frac{n\pi x}{2l}\,dx\right]$$

$$= \frac{b}{l^2}\left[-x\left(\frac{2l}{n\pi}\right)\cos\frac{n\pi x}{2l} + \left(\frac{2l}{n\pi}\right)^2 \sin\left(\frac{n\pi x}{2l}\right)\right]_0^l$$

$$+\frac{b}{l^2}\left[-(2l-x)\left(\frac{2l}{n\pi}\right)\cos\frac{n\pi x}{2l} - \left(\frac{2l}{n\pi}\right)^2 \sin\frac{n\pi x}{2l}\right]_l^{2l}$$

$$= \frac{b}{l^2}\left[-\frac{2l^2}{n\pi}\cos\frac{n\pi}{2} + \frac{4l^2}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right) + \frac{2l^2}{n\pi}\cos\frac{n\pi}{2} + \frac{4l^2}{n^2\pi^2}\sin\frac{n\pi}{2}\right]$$

$$= \frac{b}{l^2}\left(\frac{8l^2}{n^2\pi^2}\sin n\pi/2\right)$$

$$C_n = \frac{8b}{n^2\pi^2}\sin n\pi/2$$

$= 0$ if n is even

$= \frac{8b}{n^2\pi^2}\sin n\pi/2$ n is odd

$$y(x,t) = \sum_{n=odd}^{\infty}\frac{8b}{n^2\pi^2}\sin n\pi/2 \sin\frac{n\pi x}{2l}\cos\frac{n\pi a t}{2l}$$

$$= \sum_{n=1}^{\infty}\frac{8b}{(2n-1)^2\pi^2}\sin\frac{(2n-1)\pi}{2}\sin\frac{(2n-1)\pi x}{2l}\cos\frac{(2n-1)\pi a t}{2l}$$

$$= \frac{8b}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}(-1)^{n-1}\sin\frac{(2n-1)\pi x}{2l}\cos\frac{(2n-1)\pi a t}{2l}$$