TOPIC 5: PROBLEMS ON ONE DIMENSIONAL WAVE EQUATION

Problems on Vibrating String with non-zero initial velocity

The boundary and initial conditions of the deflection $y(x,t)$ are

(1) $y(0,t)=0$ (2) $y(l,t)=0$ (3) $y(x,0)=0$ (4) $\frac{\partial y}{\partial t}(x,0)=f(x)$

The suitable soln is

$$y(x,t) = (C_1 \cos px + C_2 \sin px)(C_3 \cos pat + C_4 \sin pat) \quad -(5)$$

Apply cond. (1) we get $C_1=0$

(2) $p = n\pi/l$

(3) $C_3 = 0$

The most general soln is

$$y(x,t) = \sum_{n=1}^{\infty} C_n \sin\frac{n\pi x}{l} \sin\frac{n\pi a t}{l}$$

Apply (4) cond.

$$\left(\frac{\partial y}{\partial t}\right)_{t=0} = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{l} = f(x)$$

where $B_n = C_n \frac{n\pi a}{l}$

$$B_n = b_n = \frac{2}{l}\int_0^l f(x) \sin\frac{n\pi x}{l}\, dx$$

$$\therefore C_n = \frac{l}{n\pi a} B_n$$

A tightly stretched string with fixed end points $x=0$ and $x=l$ is initially at rest in its equilibrium position. If it is set vibrating string giving each point a velocity $\lambda x(l-x)$ show that the displacement is

$$y(x,t) = \frac{8\lambda l^3}{a\pi^4} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^4} \sin\frac{(2n-1)\pi x}{l} \sin\frac{(2n-1)\pi a t}{l}$$

Sol: The wave eqn is $\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2}$

Boundary & initial conds.

(1) $y(0,t)=0 \ \forall t$

(2) $y(l,t)=0$ (3) $y(x,0)=0 \quad 0<x<l$

(4) $\left(\frac{\partial y}{\partial t}\right)_{(x,0)} = \lambda x(l-x) \quad 0<x<l$

The suitable sol. is

$y(x,t) = (C_1 \cos px + C_2 \sin px)(C_3 \cos pat + C_4 \sin pat)$ — (I)

Apply cond. (1) in I

$y(0,t) = C_1 (C_3 \cos pat + C_4 \sin pat) = 0$

$C_3 \cos pat + C_4 \sin pat \neq 0$

$\therefore \boxed{C_1 = 0}$

Substitute $C_1 = 0$ in (I)

$y(x,t) = C_2 \sin px (C_3 \cos pat + C_4 \sin pat)$ — (II)

Apply cond. (2) in II

$y(l,t) = C_2 \sin pl (C_3 \cos pat + C_4 \sin pat) = 0$

Here $C_3 \cos pat + C_4 \sin pat \neq 0$

$C_2 \sin pl$; either $C_2 = 0$ (or) $\sin pl = 0$

Suppose $C_2 = 0$ already we have $C_1 = 0$

$\therefore C_2 \neq 0$. $\sin pl = 0 = \sin n\pi$

$pl = n\pi \Rightarrow p = n\pi / l$.

Now sub $p = n\pi/l$ in eq. II

$y(x,t) = C_2 \sin \frac{n\pi x}{l} \left(C_3 \cos \frac{n\pi at}{l} + C_4 \sin \frac{n\pi at}{l}\right)$ — (III)

Apply cond. (3) in III

$y(x,0) = C_2 \sin \frac{n\pi x}{l} C_3 = 0$

$$C_2 C_3 \sin\frac{n\pi a}{l} = 0$$

$\sin\frac{n\pi x}{l} \neq 0$ (It is defined for all x)

$C_2 \neq 0$

$\therefore C_3 = 0$.

Sub $C_3 = 0$ in eq (B)

$$y(x,t) = C_2 C_4 \sin\frac{n\pi x}{l} \sin\frac{n\pi a t}{l}$$

$$= C_n \sin\frac{n\pi x}{l} \sin\frac{n\pi a t}{l} \quad \text{— (IV)}$$

The most general soln.

$$y(x,t) = \sum_{n=1}^{\infty} C_n \sin\frac{n\pi x}{l} \sin\frac{n\pi a t}{l} \quad \text{— (V)}$$

Before apply condn (iv) diff (V)

$$\left(\frac{\partial y}{\partial t}\right)_{(x,t)} = \sum_{n=1}^{\infty} C_n \sin\frac{n\pi x}{l}\left(\frac{n\pi a}{l}\right)\cos\frac{n\pi a t}{l}$$

Apply (iv) condn

$$\frac{\partial y}{\partial t} = \sum_{n=1}^{\infty} C_n \sin\frac{n\pi x}{l}\left(\frac{n\pi a}{l}\right) = \lambda x(l-x)$$

$$= \sum_{n=1}^{\infty}\left(C_n \frac{n\pi a}{l}\right)\sin\frac{n\pi x}{l} = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{l}$$

$$B_n = C_n \frac{n\pi a}{l}$$

To find B_n:- Expand $\lambda x(l-x)$ in a half range sine series

$$\lambda x(l-x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

$$b_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\,dx.$$

$$B_n = \frac{2}{l}\int_0^l \lambda x(l-x)\sin\frac{n\pi x}{l}\,dx.$$

$$= \frac{2\lambda}{l}\left[-(2x-l)\left(\frac{l}{n\pi}\right)\cos\frac{n\pi x}{l} + (2-2x)\left(\frac{l}{n\pi}\right)^2 \sin\frac{n\pi x}{l} - 2\left(\frac{l}{n\pi}\right)^3\cos\left(\frac{n\pi x}{l}\right)\right]_0^l$$

$$= \frac{2\lambda}{l}\left[\left(-2\left(\frac{l}{n\pi}\right)^3(-1)^n\right) - \left(-2\left(\frac{l}{n\pi}\right)^3\right)\right]$$

$$= \frac{2\lambda}{l}\left[-2\left(\frac{l}{n\pi}\right)^3(-1)^n + 2\left(\frac{l}{n\pi}\right)^3\right]$$

$$= \frac{2\lambda}{l}\times 2\left(\frac{l}{n\pi}\right)^3\left[1-(-1)^n\right]$$

$$= \frac{4\lambda l^2}{n^3\pi^3}\left(1-(-1)^n\right) = C_n\left(\frac{n\pi a}{l}\right)$$

$$C_n = \left(\frac{l}{n\pi a}\right)\left(\frac{4\lambda l^2}{n^3\pi^3}\right)\left(1-(-1)^n\right)$$

$$C_n = \begin{cases} 0 & n = \text{even} \\ \dfrac{8\lambda l^3}{a n^4\pi^4} & n = \text{odd} \end{cases}$$

$$y(x,t) = \frac{8\lambda l^3}{a\pi^4}\sum_{n\,\text{odd}}^{\infty}\frac{1}{n^4}\sin\frac{n\pi x}{l}\sin\frac{n\pi at}{l}$$

$$= \frac{8\lambda l^3}{a\pi^4}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^4}\sin\frac{(2n-1)\pi x}{l}\sin\frac{(2n-1)\pi at}{l}$$

— x —

(2) A tightly stretched string of length l is initially at rest in its equilibrium position and each of its pts is given the velocity $V_0 \sin^3\left(\frac{\pi x}{l}\right)$. Find the displacement.

Soln: The wave eqn. is $a^2\frac{\partial^2 y}{\partial x^2} = \frac{\partial^2 y}{\partial t^2}$

The conditions are (1) $y(0,t)=0$ (2) $y(l,t)=0$

(3) $y(x,0)=0$ (4) $\left(\frac{\partial y}{\partial t}\right)_{(x,0)} = V_0\sin^3\left(\frac{\pi x}{l}\right)$

$$\frac{3v_0}{4}\sin\left(\frac{\pi x}{l}\right) - \frac{v_0}{4}\sin\left(\frac{3\pi x}{l}\right) = \sum_{n=1}^{\infty} B_n\left(\frac{n\pi a}{l}\right)\sin\left(\frac{n\pi x}{l}\right)$$

$$= B_1\left(\frac{\pi a}{l}\right)\sin\left(\frac{\pi x}{l}\right) + B_2\sin\left(\frac{2\pi x}{l}\right)\left(\frac{2\pi a}{l}\right) + B_3\left(\frac{3\pi a}{l}\right)\sin\left(\frac{3\pi x}{l}\right) + \cdots$$

$$B_1 = \frac{3v_0}{4} \qquad B_3 = -\frac{v_0}{4}$$

$$B_1 \left(\frac{\pi a}{l}\right) \Rightarrow \frac{3v_0}{4}\left(\frac{l}{\pi a}\right) = C_1$$

$$\frac{3v_0 l}{4\pi a} = C_1$$

$$B_3 = C_3\left(\frac{3\pi a}{l}\right) = -\frac{v_0}{4}$$

$$C_3 = -\frac{l v_0}{12\pi a}$$

$$y(x,t) = \frac{3v_0 l}{4\pi a}\sin\left(\frac{\pi x}{l}\right)\sin\left(\frac{\pi a t}{l}\right) - \frac{v_0 l}{12\pi a}\sin\left(\frac{3\pi x}{l}\right)\sin\left(\frac{3\pi a t}{l}\right)$$

(8) A string of length l is initially at rest in the equilibrium position and motion is started by giving each of its pts a velocity given by

$$v = \begin{cases} cx & \text{if } 0 \le x \le l/2 \\ c(l-x) & l/2 \le x \le l \end{cases}$$

Find the displacement.

Soln: The wave eqn.

Boundary Condn.

(1) $y(0,t) = 0$ $\quad$ $y(l,t) = 0$ $\quad$ $y(x,0) = 0$

(2) $\left(\frac{\partial y}{\partial t}\right)_{t=0} = \begin{cases} cx & 0 \le x \le l/2 \\ c(l-x) & l/2 \le x \le l \end{cases}$

$y(x,t) = (C_1\cos px + C_2\sin px)(C_3\cos pat + C_4\sin pat)$

Apply Cond (4)

$$\left(\frac{\partial y}{\partial t}\right)_{t=0} = \sum \frac{n\pi a}{l} B_n \sin\left(\frac{n\pi x}{l}\right) = \begin{cases} cx & 0 \le x \le l/2 \\ c(l-x) & l/2 \le x \le l \end{cases}$$

$$B_n = C_n\left(\frac{n\pi a}{l}\right)$$

To find B_n

$$b_n = \frac{2}{l}\int_0^l f(x)\sin\left(\frac{n\pi x}{l}\right)dx \qquad (6)\ \&\ (7)\ \ b_n = C_n$$

$$= \frac{2c}{l}\left[\int_0^{l/2} x \sin\frac{n\pi x}{l}\,dx + \int_{l/2}^{l}(l-x)\sin\frac{n\pi x}{l}\,dx\right]$$

$$= \frac{2c}{l}\left\{\left[-x\left(\frac{l}{n\pi}\right)\cos\left(\frac{n\pi x}{l}\right) + \left(\frac{l}{n\pi}\right)^2\sin\left(\frac{n\pi x}{l}\right)\right]_0^{l/2} + \left[-(l-x)\left(\frac{l}{n\pi}\right)\cos\left(\frac{n\pi x}{l}\right) - \left(\frac{l}{n\pi}\right)^2\sin\left(\frac{n\pi x}{l}\right)\right]_{l/2}^{l}\right\}$$

$$= \frac{2c}{l}\left[-\frac{l^2}{2n\pi}\cos(n\pi/2) + \left(\frac{l}{n\pi}\right)^2\sin(n\pi/2) + \frac{l^2}{2n\pi}\cos(n\pi/2) + \left(\frac{l}{n\pi}\right)^2\sin(n\pi/2)\right]$$

$$= \frac{2c}{l}\left(\frac{2l^2}{n^2\pi^2}\right)\sin(n\pi/2) = \frac{4lc}{n^2\pi^2}\sin(n\pi/2)$$

$$C_n = \frac{l}{n\pi a}B_n = \frac{l}{n\pi a}\left(\frac{4lc}{n^2\pi^2}\sin(n\pi/2)\right)$$

$$y(x,t) = \sum_{n=1}^{\infty}\frac{4l^2c}{n^3\pi^3 a}\sin(n\pi/2)\cdot\sin\left(\frac{n\pi x}{l}\right)\sin\left(\frac{n\pi a t}{l}\right)$$